THIRD-YEAR MATHEMATICS

BRESLICH

THE UNIVERSITY OF CHICAGO
MATHEMATICAL SERIES

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THE SCHOOL OF EDUCATION
TEXTS AND MANUALS

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Breslich

THIRD-YEAR MATHEMATICS *for* SECONDARY SCHOOLS

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

THE BAKER & TAYLOR COMPANY
NEW YORK

THE J. K. GILL COMPANY
PORTLAND

THE CUNNINGHAM, CURTISS & WELCH COMPANY
LOS ANGELES

THE CAMBRIDGE UNIVERSITY PRESS
LONDON AND EDINBURGH

THE MARUZEN-KABUSHIKI-KAISHA
TOKYO, OSAKA, KYOTO, FUKUOKA, SENDAI

THE MISSION BOOK COMPANY
SHANGHAI

GOTTFRIED WILHELM LEIBNITZ

GOTTFRIED WILHELM LEIBNITZ was born at Leipzig in 1646 and died at Hanover in 1716. His was the genius that was both precocious and universal. Even from early childhood he overcame the most trying obstacles. He proved himself of extraordinary qualities before he was twelve. Though he had mastered the ordinary texts in mathematics, philosophy, theology, and law before he was twenty, not until he reached the age of twenty-six, when he was sent on a political errand to Paris where he made the acquaintance of Huygens, did he become strongly interested in mathematics. He achieved eminence of the highest order in mathematics, philosophy, theology, law, and languages. For forty years following 1676 he held the post of librarian of the house of Brunswick and Hanover, earning the highest honors and distinctions in the services of his house only to be cast aside in old age by the existing head of the Brunswick family, when he became George I of England.

Leibnitz' political and religious papers touching affairs of the dynasty from 1673–1713 constitute an indispensable contribution to the history of his time.

His place in the history of philosophy is even larger than it is in mathematics. The Leibnitzian system of philosophy constitutes a most important epoch in the history of philosophical doctrines. Leibnitz' life denies the prevailing idea that mathematical genius is necessarily narrow and specialized. It also furnishes a conspicuous example of a most important contributor to the advance of mathematics who was not by profession, at any time during his life, a teacher of the science.

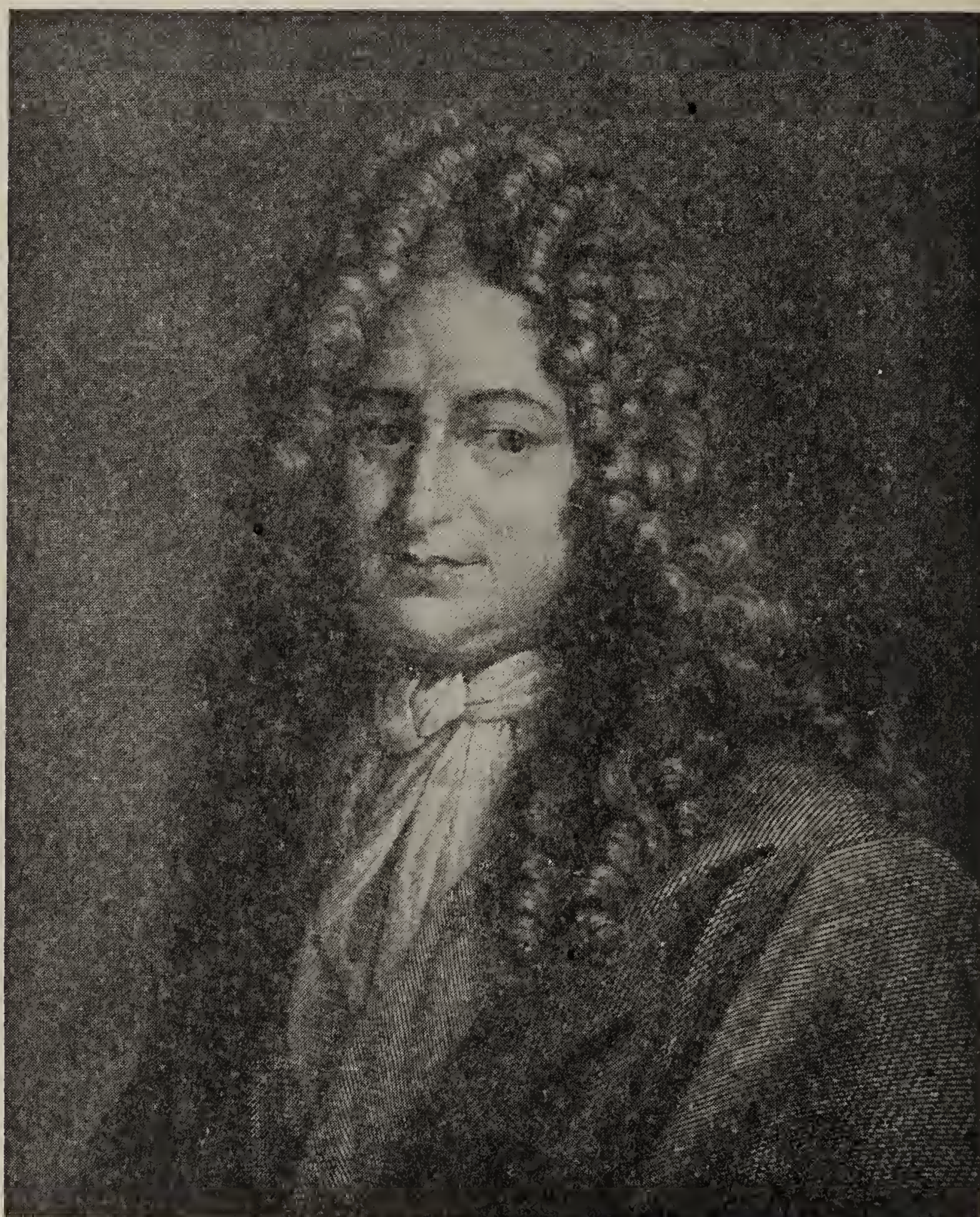
His chief services to mathematical science consist of his independent invention of the language, if not the substance, of the differential calculus, his work on osculating curves, his fundamental work on the theory of envelopes, his explanation of the method of expansion of functions by indeterminate coefficients, and his recognition of the theory of determinants and his developmental work on the theory.

His work displays great skill in analysis, but like the work of most geniuses it is unfinished and characterized by frequent errors. But he blazed out many new routes through the mathematical regions which men of lesser genius were aided later in converting into comfortable highways and thoroughfares.

The later years of his life were embittered by a contest with the over-zealous friends of Newton over the question of priority of invention of the calculus as between him and Newton. Subsequent times have seemed to settle the controversy on the basis that Leibnitz and Newton were independent inventors of the calculus, and that most certainly the modern notation of the calculus is due to Leibnitz. For more detail about this famous controversy any history of mathematics may be consulted.

In character Leibnitz was quick-tempered, intolerant, selfish, and inordinately conceited. But the products of his genius will ever adorn and enrich the pages of mathematical and philosophical history.

[See Ball's or Cajori's or Tropfke's *History*.]



GOTTFRIED WILHELM LEIBNITZ

Third-Year Mathematics *for* Secondary Schools

BY

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THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

Published September 1917

Composed and Printed By
The University of Chicago Press
Chicago, Illinois, U.S.A.

EDITOR'S PREFACE

This third unit of Mr. Breslich's course in general mathematics for high schools aims primarily to carry forward the spirit and the method of the two former volumes. By using chapter xv at the beginning of the year as a syllabus for reviewing the ground covered in the previous work in geometry, this volume may be readily taken up by classes whose previous work has been in standard courses of algebra and geometry.

To aid pupils who do not go beyond the high school, and whose first two years of training have been in correlated mathematics, to become sufficiently familiar with the standard special mathematical methods and principles of algebra, geometry, and trigonometry, as such, and to command the existing literature of these branches, the type of correlation here used is what may be termed *topical.* A certain type of subject-matter is allowed emphasis for a sufficient time to enable pupils to master the appropriate type of methodology. This will seem to a superficial critic a departure from the close type of correlation of the two preceding texts. This variation is, however, intentional, to the end that algebra, geometry, and trigonometry, as such, may be firmly grasped by the pupil while he is yet in high school. The correct description of the prevailing procedure here is: *Isolation in details; but correlation in major matters.* This assures any real benefits of an isolated type of treatment without losing the more important values of correlation. Mathematical training must foster both *concentration* and *generalship.* Classroom experience has verified the propriety of this type of correlation for third-year classes.

The author would request open-minded teachers to give the form of reconstructed mathematics herewith presented a fair classroom test. He will gladly accept the issue of such a test. Superintendents and principals should feel that here is something of the sort their spokesmen have been urging, and see to it that the text be given a fair test. Better things can hardly be obtained except through the testing of different methods. Of the methods deserving of a classroom test, those that have proved successful in particular instances are most worth while. The material of this volume belongs to this class. May its friends become as numerous as are those of its companion volumes!

G. W. MYERS

CHICAGO, ILL.
August, 1917

AUTHOR'S PREFACE

This book is the third of the series of textbooks on secondary mathematics. It is designed primarily as a third unit of a year's work to follow the first two unit-courses worked out by the author in *First-Year Mathematics* and *Second-Year Mathematics.* It completes the study of high-school algebra, trigonometry, and solid geometry.

In accordance with the general plan of the series, the book aims to teach in combination mathematical topics which are naturally closely related to each other even though drawn from separate mathematical subjects.

Such an arrangement has the advantage of developing the subject of secondary-school mathematics in a sequence which is both psychological and logical. Indeed, the student's understanding of the meaning and the utility of the subject is deepened to such an extent that he is better able to appreciate the scientific character of mathematics than when he is studying the separate subjects. The result is that he is more disposed to continue the study of mathematics.

Through proper correlation the whole third-year work can be better motivated and becomes more concrete, each subject gaining from the study of others. For example, the student appreciates the need of studying the theory of logarithms because of their usefulness as a tool for solving problems in trigonometry. He further sees that he must master the theory of exponents in order that he may understand the fundamental principles of the theory of logarithms.

In the study of simultaneous linear and quadratic equations, in connection with intersecting straight lines and curves, the abstract processes of solving the various types of systems of equations are represented concretely and are easily understood and remembered.

Through the graphical representation of equations of the form $y=f(x)$, where $f(x)$ is either a polynomial or a trigonometric function, he learns to appreciate the fundamental concept of functional correspondence, which is of greatest value to him as a natural introduction to *analytical geometry*.

Thus correlation removes the disadvantages of the topical plan without losing its advantages. It arouses and holds the student's interest. Many students who find third-year algebra as a separate subject too abstract and uninteresting take great delight in a third-year course in which algebra and trigonometry are correlated, because they enjoy the study of trigonometry and its applications.

Special attention is directed to the following features of this course:

1. Reviews are carried on at frequent intervals throughout the course. Hence the introductory review found in most third-year texts has been omitted. Since the course begins with new work, the student has at once the exhilaration of taking a step in advance, while at the same time he is reviewing in a way that gives him a new and higher view of the subject-matter. His time and effort are economized through the avoidance of useless repetitions.

2. At the end of each chapter the principal facts are summarized.

3. The last chapter of the book is a syllabus of all the theorems of plane and solid geometry which were studied

in the preceding courses. This may be used for reference purposes while the student is taking the course. It will also be found effective as a basis for a final review of plane and solid geometry as such. The list, at the end of the book, which gives all important mathematical formulas so far studied serves the same purpose.

4. The material has been carefully selected with a view to stimulating independent thinking and to preparing the student for collegiate mathematics. The number of supplementary exercises is sufficiently large to furnish the drill needed to enable the student to meet the requirements of college entrance examinations. For this purpose many problems taken from entrance examinations of various colleges have been included.

5. The historical notes distributed throughout the book add to the interest of the student in genetic phases of the work. The portraits appearing as inserts have been taken from the *Philosophical Portrait Series*, published by the Open Court Publishing Company, Chicago.

The author takes great pleasure in acknowledging his indebtedness to his colleagues in the department of mathematics for many valuable constructive criticisms. He is indebted also to Principal F. W. Johnson, of the University High School, and to Professor Charles H. Judd, Director of the School of Education, for sympathetic appreciation and encouragement during the preparation of the manuscript.

Ernst R. Breslich

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STUDY HELPS FOR STUDENTS[1]

The habits of study formed in school are of greater importance than the subjects mastered. The following suggestions, if carefully followed, will help you make your mind an efficient tool. Your daily aim should be to learn your lesson in *less* time, or to learn it *better* in the *same* time.

1. Make out a definite daily program, arranging for a definite time for the study of mathematics. You will thus form the habit of concentrating your thoughts on the subject at that time.

2. Provide yourself with the material the lesson requires; have on hand textbook, notebook, ruler, compass, special paper needed, etc. When writing, be sure to have the light from the left side.

3. Understand the lesson assignment. Learn to take notes on the suggestions given by the teacher when the lesson is assigned. Take down accurately the assignment and any references given. Pick out the important topics of the lesson before beginning your study.

4. Learn to use your textbook, as it will help you to use other books. Therefore understand the purpose of such devices as index, footnotes, etc., and use them freely.

5. Do not lose time getting ready for study. Sit down and begin to work at once. Concentrate on your work, i.e., put your mind on it and let nothing disturb you. Have the will to learn.

[1] These study helps are taken from *Study Helps for Students in the University High School.* They have been found to be very valuable to students in *learning* how to study and to teachers in *training* students how to study effectively.

6. As a rule it is best to go over the lesson quickly, then to go over it again carefully; e.g., before beginning to solve a problem read it through and be sure you understand what is given and what is to be proved. Keep these two things clearly in mind while you are working on the problem.

7. Do individual study. Learn to form your own judgments, to work your own problems. Individual study is honest study.

8. Try to put the facts you are learning into practical use if possible. Apply them to present-day conditions. Illustrate them in terms familiar to you.

9. Take an interest in the subject. Read the corresponding literature in your school library. Talk to your parents about your school work. Discuss with them points that interest you.

10. Review your lessons frequently. If there were points you did not understand, the review will help you to master them.

11. Prepare each lesson every day. The habit of meeting each requirement punctually is of extreme importance.

CHAPTER I

FUNCTIONS. EQUATIONS IN ONE UNKNOWN

Function. Variable. Constant

1. The formula $i = prt$ may be used to compute the interest, i, of a principal, p, at the rate, r, for t years. If the rate and principal remain the same, the interest, i, depends upon the time, t, in the sense that if one is changed, the other changes correspondingly. The time and interest are *variables,* the principal and rate *constants,* and the interest is said to be a *function* of the time.

Dependence of one magnitude upon another is met frequently. For example, the premium of a life insurance policy depends upon the age of the applicant, the distance passed over by a moving body depends upon the time, the length of a circle depends upon the radius.

Sometimes this dependence is expressed in the form of an equation. Thus, the length of a circle is given by the equation $c = 2\pi r$. Because of this equation, to every value of c there corresponds a definite value of r. This is often expressed by saying that c is a *function* of r. The symbols c and r are *variables,* π is a *constant.*

2. Constant. A symbol which represents the same number throughout a discussion, or in a problem, is a **constant.**

EXERCISE

In the equations, $A = \pi r^2$, $A = \frac{1}{2}bh$, $d = rt$, $s = \frac{1}{2}gt^2$, $v = \frac{4}{3}\pi r^3$*, one letter depends upon one or more other letters for its value. Which symbols in these equations are constants?

* When such forms as r^2, $\frac{1}{2}bh$, rt, $16t^2$, $2x+3$ and $\sqrt{x^2-25}$, first came into mathematics, they were regarded merely as shortened

3. Variable. A **variable** is a symbol representing *different* numbers in a problem.

Name the variables in the foregoing equations.

4. Function. If two variables x and y are so related that to every value of x there corresponds a definite value of y, then y is said to be a **function** of x.

EXERCISES

In the following relations show that one symbol is a function of one or more others.

1. $s=\frac{1}{2}(a+b+c)$
2. $A=\frac{1}{2}cr$
3. $A=\frac{a^2}{4}\sqrt{3}$
4. $V=\frac{1}{3}bh$
5. $V=\frac{\pi r^2h}{3}$
6. $C=\frac{5}{9}(F-32)$
7. Name the constants and the variables in exercises 1–6.

5. Functional notation. Sometimes we are interested in the *relation** between two variables rather than in the variables themselves. For example, in *uniform motion* the distance is equal to the rate multiplied by the time, but with *falling bodies* the distance is approximately 16 multiplied by the square of the time. The first of these laws is expressed in the form $d=rt$, the second in the form $d=16t^2$. In these two cases d is *not the same* function of t.

---

statements of rules of reckoning, just as in percentage we regarded $p=b\cdot r$ as a short way of stating: *percentage equals base times rate.*

Later these forms came to be regarded as either (1) rules of reckoning, or (2) the results of the reckoning. As results, they were regarded as numbers, and could be added, subtracted, multiplied, and divided. This gave rise at once to modern algebra.

* *Relation* means any interdependence, and not necessarily ratio.

The equation $f=2n$ expresses the relation between the number of miles and the railroad fare at 2 cents a mile. The relation between the number of eggs and the cost at 2 cents each is represented by the equation $c=2n$. In these two examples f and c are the *same* function of n.

The symbol $f(x)$ is used to represent a function of x. It is read *function of x*, or briefly, *f of x*. To distinguish between different functions of x other letters are used, as in $g(x)$ or $F(x)$.*

Thus, if the function $3x+2$ is denoted by $f(x)$ and if, in the same discussion, we wish to refer briefly to some other function, as $\sqrt{16-x^2}$, we may denote the latter by the symbol $F(x)$.

6. Evaluation† of functions. To find the value of a function as x^2+5x+3, for a given value of x, the variable x in the function is replaced by the given value. If x^2+5x+3 is denoted by $f(x)$, then $3^2+5\cdot 3+3$ is denoted

* The word "function" was first used in a mathematical sense in 1694 by Leibnitz, though in a different sense from that given here. In October of the same year James Bernoulli employed the word in the Leibnitzian sense. John Bernoulli employed the word in its modern sense in a letter to Leibnitz in June, 1698. Leibnitz' answer at the end of July of the same year shows that he too had given the word "function" its present meaning. The new technical term was first employed in print in a pamphlet by John Bernoulli in 1706. The latter was also the first to define the word. This he did in the Reports of the French Academy of 1718.

John Bernoulli and Leibnitz both used special symbols for "function" in 1698. Neither used the symbol defined here. Euler, in a scientific publication of 1734–35, was the first to use the letter f, followed by the variable inside of parentheses, as the symbol for *function*. The French mathematician Clairaut was at the same time using a symbol in which the f of Euler's symbol was replaced by a Greek capital letter (Tropfke, *Geschichte der Elementar-Mathematik*, Band I, S. 142–43).

† *Evaluation* means to find the value.

by $f(3)$. Thus, $f(3)$ is the result obtained by replacing x in $f(x)$ by 3, or the *functional value* of the function $f(x)$ for the particular value $x=3$ of the variable x.

EXERCISES

1. If $f(x)=x^2+3x+5$, find $f(2)$, $f(0)$, $f(-1)$, $f(a)$.

$$f(2)=2^2+3\cdot 2+5=15$$
$$f(0)=0^2+3\cdot 0+5=5$$
$$f(-1)=(-1)^2+3(-1)+5=7$$
$$f(a)=a^2+3a+5$$

2. If $f(x)=x^2-4x+3$ and $F(x)=2x^2-5$, find $F(6)-f(2)$.

$$F(6)=2\cdot 6^2-5=67$$
$$f(2)=2^2-4\cdot 2+3=-1$$
$$\therefore F(6)-f(2)=68$$

3. If $f(y)=y^3-3y^2+7y-1$, find $f(1)$, $f(-2)$, $f(0)$.

4. If $f(r)=mr^2+nr+p$, find $f(-3)$, $f(\frac{1}{2})$, $f(a)$.

5. If $f(x)=x^2+2x+5$ and $g(x)=x^2-3x+2$, find $f(2)+g(-1)$; find $\dfrac{f(4)}{g(3)}$.

Linear Function

7. The function $ax+b$. Functions like $2x+5$, $\frac{1}{2}x-7$, $3x+\frac{1}{4}$ are of the form $ax+b$, where a and b are constants and x is a variable.

EXERCISES

1. Which of the following functions are of the form $ax+b$?

$2\pi r$; $\frac{9}{5}C+32$; $2x^2-4$; $30t$; v_0+gt; $\frac{1}{x}$; $\frac{5}{9}(F-32)$; $\cos x$; 5^x.

2. Give other examples of functions of the form $ax+b$.

8. Graph* of the function $ax+b$. The relation between the variable x and the function $ax+b$ may be represented graphically.

* Graphing was introduced as a systematic mathematical method by Descartes in 1637.

For example, a boy, on a certain date, deposits in a bank the sum of \$3. He then deposits \$2 regularly at the end of every week. How much money will he have in x weeks?

Show that $F(x)=2x+3$.

Find a number of corresponding values of x and $F(x)$ and tabulate them as in Fig. 1. Plotting the values of the variable x as *abscissas* and the values of the function $F(x)$ as *ordinates*, we obtain the straight line AB.

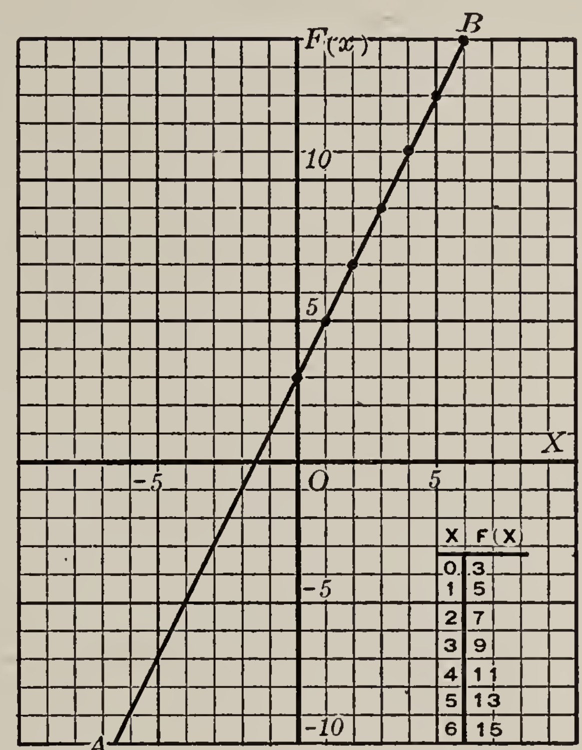


FIG. 1

9. Linear function. The function $ax+b$ is a function of the *first degree* in x. It is also called a **linear function** of x, because the graph is a straight line.

The following shows that, in general, a straight line in a plane can be represented by an equation of the form $f(x)=mx+b$:

1. Let P be any point on the straight line ABC, not passing through the origin O, Fig. 2.

Then $OQ=x$ and $PQ=f(x)$.

Denote the distances AO and BO by a and b, respectively.

Draw $BD\perp PQ$ and denote angle DBP by the letter s.

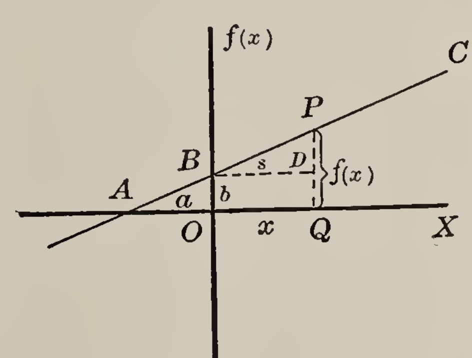


FIG. 2

Show that $\tan s=\dfrac{DP}{BD}=\dfrac{f(x)-b}{x}$

Denoting the value of tan s by m, we have

$$m=\frac{f(x)-b}{x}$$

$$\therefore f(x)=mx+b$$

2. When AB passes through O, Fig. 3,

$$\tan s=\frac{f(x)}{x}, \text{ or } m=\frac{f(x)}{x}$$

$$\therefore f(x)=mx$$

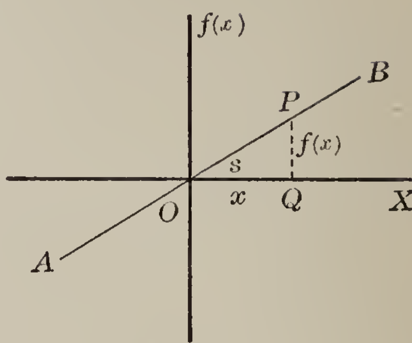


Fig. 3

3. When a line is parallel to the x-axis or to the $f(x)$-axis, show that its equation is of the form $f(x)=c$, or of the form $x=c$, respectively.

10. Intercepts. The numbers a and b, Fig. 2, are the **intercepts** made by the line AC on the x-axis and the $F(x)$-axis, respectively.

EXERCISES

Construct the graphs of the linear functions in the following examples:

1. The length of a circle is approximately equal to 3.14 multiplied by the diameter, i.e., $c=3.14d$.

2. The temperature in Fahrenheit degrees is 32 degrees greater than $\frac{9}{5}$ of the temperature in Centigrade degrees, i.e., $F=\frac{9}{5}C+32$.

3. When a body falls from rest, the velocity, v, at any time, t, is given by the equation $v=gt$, where $g=32$, approximately.

4. Graph the equations $f(x)=2x+3$; $f(x)=-4$; $x=2$.

Direct Variation

11. Direct variation. In preceding work (§ 200, *Second-Year Mathematics*) we have seen that the statements *y varies as x, y is directly proportional to x*, or *y*

varies directly as x are expressed algebraically by means of the equation $y=cx$. Thus, the statements: A man's pay is directly proportional to the number of days he works, the distance of a body moving at a uniform rate varies directly as the time, the length of a circle varies as the radius, are written respectively $p=ct$, $d=ct$, $l=cr$. The constant c is the **constant of variation.** The variables p, d, and l are *linear functions* of t, t, and r, respectively.

EXERCISES

Express the following statements in the form of equations:

1. The area of a sphere varies as the square of the radius.

2. The volume of a cylinder varies directly as the square of the diameter of the base if the altitude remains constant.

3. The area of an equilateral triangle varies as the square of a side.

4. When a body falls from rest (in a vacuum), the velocity varies directly as the time of falling.

5. When a spring is stretched by a force, f, the distance the spring is stretched (elongation) varies as the force (Hooke's law).

6. The average consumption of coal for steam boilers varies directly as the number of square feet of grate surface.

7. The diagonal of a cube varies as the edge. The edge is 5 when the diagonal is 8.5. Find the diagonal when the edge is 10.

1. Show that $d=c\cdot e$.
2. Determine the constant of variation:

   Since $d=c\cdot e$, $8.5=c\cdot 5$; $\therefore\ c=\dfrac{8.5}{5}=1.7$.

3. The diagonal may now be determined: $d=(1.7)(10)=17$.

8. Represent graphically the change in the diagonal, exercise 7, as the edge changes.

9. The area of a circle varies as the square of the radius and the area is 113 sq. ft. when the radius is 6 feet. Find the area of a circle whose radius is $2\frac{1}{2}$ feet.

10. The time required by a pendulum to make one vibration varies as the square of the length. A pendulum 100 cm. long vibrates once in a second. What is the time of vibration of a pendulum 49 cm. long?

11. The weight of a liquid is directly proportional to the volume. If 10 cu. ft. of water weigh 625 lb., what is the weight of 25 cubic feet?

12. The pressure in pounds per square inch of a column of water varies directly as the height of the column in feet. A column of water 2.5 ft. high exerts a pressure of 1.08 lb. per square inch. Find the height of a column exerting 1.84 pounds.

Quadratic Function

12. Quadratic function. The functions πr^2, $s_0+\frac{1}{2}gt^2$, $3x^2-4x+2$ are of the *second degree.* Show that they are of the form ax^2+bx+c. Functions of the form ax^2+bx+c in which $a\neq 0$ are called **quadratic functions.**

EXERCISES

Show that the following functions are of the form ax^2+bx+c and determine in each case the values of a, b, and c.

1. $2+4x^2-x$

2. $\frac{1}{2}+\frac{3}{5}x-x^2-5$

3. $ax^2-mx+2x^2-12$

4. x^2-2

5. $\frac{x^2}{8}$

6. $4\sin^2 x+3\sin x-7$

13. Graph of the function ax^2+bx+c. In § 9 it was shown that the graph of a *linear* function is a *straight line.* It will be seen that the graph of the *quadratic* function ax^2+bx+c is a smooth *curve* no three points of which lie on the same straight line.

EXERCISES

1. Graph the function $f(x) = x^2$.

Tabulate values of $f(x)$ corresponding to values of x between -3 and $+3$ (Fig. 4).

By plotting the points corresponding to the pairs of numbers in the table and drawing the curve passing through these points, the graph of x^2 is obtained. This curve is called a **parabola**.

x	$f(x)$
-3	9
-2	4
-1	1
0	0
$+1$	1
$+2$	4
$+3$	9

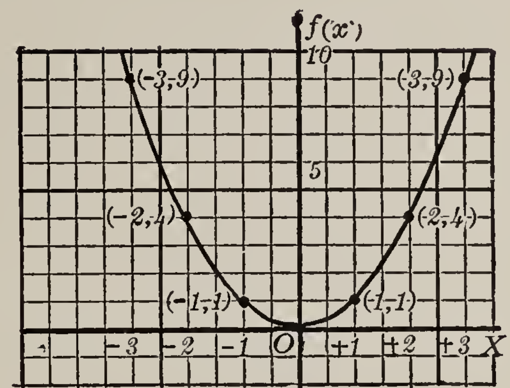


Fig. 4

If x is a negative number and very large, $f(x)$ is positive and very large, e.g., for $x = -1{,}000$, $f(x) = +1{,}000{,}000$. As x increases, remaining negative, $f(x)$ decreases. As x approaches zero, $f(x)$ also approaches zero. As x continues to increase indefinitely, $f(x)$ increases without bound. This may be represented in a table as follows:

x	$-\infty$*	negative, increasing	0	positive, increasing	$+\infty$
$f(x)$	$+\infty$†	positive, decreasing	0	positive, increasing	$+\infty$

* The symbol $-\infty$ means *increasing without bound* in the *negative* direction.

† The symbol $+\infty$ means *increasing without bound* in the *positive* direction.

The value $x = 0$ is said to be a **zero of $f(x)$**, i.e., it is a value of x such that the corresponding value of $f(x)$ is zero.

2. Graph the positive side of the curve $f(x) = x^2$, using on the x-axis a unit equal to 2 cm., and on the $f(x)$-axis a unit equal to $\frac{1}{5}$ centimeter.

Show how this graph may be used as a device for finding square roots of numbers.

3. Show that the graph of the function $f(x) = x^2$ is *symmetric* with respect to the $f(x)$-axis.

Show that the $f(x)$-axis is the perpendicular bisector of the line-segment joining any two points on the graph which have equal ordinates.

4. Graph the function πr^2, taking $\pi = 3.14$.

5. The velocity of a ball thrown vertically upward with an initial velocity of 64 ft. per second is given by the formula $v^2 = 64^2 - 64h$, where h is the height attained at any time. Show that h is a quadratic function of v. Represent graphically the function $h = f(v)$, for values of v varying from 64 to 0.

x	$f(x)$
-3	12
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5
5	12

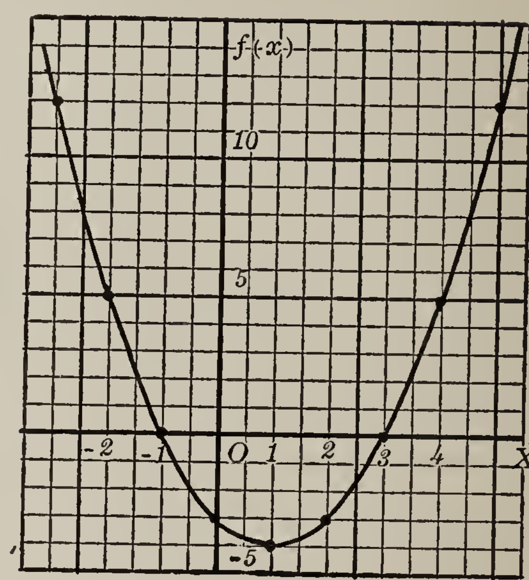


Fig. 5

6. Graph the function

$$f(x) = x^2 - 2x - 3.$$

Compute the values of $f(x)$ as in the table, Fig. 5.

Plot the points, as in Fig. 5, and draw the graph.

What are the zeros of the function $f(x) = x^2 - 2x - 3$?

Show how the graph may be used to solve the equation $x^2 - 3x - 3 = 0$.

Find the axis of symmetry of the function $x^2 - 2x - 3$.

Graph the following functions and in each case locate the axis of symmetry:

7. $x^2 - 6x + 5$

8. $3x^2 - 11x - 4$

9. $x^2 + 4x$

10. $-x^2 + 6x - 5$

Solve graphically the following quadratic equations:

11. $x^2 + 4x + 2 = 0$

12. $x^2 - 5x + 4 = 0$

13. $x^2 + x - 6 = 0$

14. $4 - 5x - x^2 = 0$

Graphical Solution of Equations of Degree Higher than the Second

14. Graph of a cubic function. Functions like $x^3 - x$, $x^3 - 6x^2 + 11x - 6$, $2x^3 - 3x + 2$ are functions of the *third degree* or *cubic functions.*

EXERCISES

1. Graph the function $f(x)=x^3-x$ and locate the zeros.

Compute the table, Fig. 6. Plot the points corresponding to the pairs of numbers in the table. Draw the curve passing through these points.

What are the zeros of x^3-x? State how the graph may be used to solve the equation $x^3-x=0$.

x	$f(x)$
-3	-24
-2	-6
-1	0
$-\frac{1}{2}$	$-\frac{3}{8}$
0	0
$\frac{1}{2}$	$\frac{3}{8}$
1	0
2	6
3	24

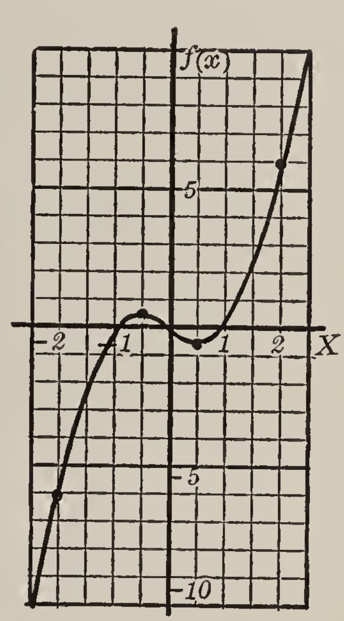


FIG. 6

2. Graph the function

$$f(x)=x^3-6x^2+11x-6$$

and use the graph to illustrate the roots of the equation $x^3-6x^2+11x-6=0$.

3. The volume, v, of a sphere is $\frac{1}{6}\pi d^3$. Show by means of a graph the changes of the volume as the diameter changes.

15. Graphical solution of equations of degree higher than the second. Exercises 1 and 2, § 14, indicate how the graph may be used to solve some *cubic* equations. Equations of degree *higher* than the third may be solved in a similar way.

EXERCISES

Solve the following equations, giving the values of the roots approximately to the first decimal place:

x	$f(x)$
-4	-162
-3	0
-2	$+40$
-1	$+18$
0	-6
$\frac{1}{2}$	0
1	28
2	180

1. $10x^3+29x^2-5x-6=0$

Graph the function

$$f(x)=10x^3+29x^2-5x-6,$$

Fig. 7, and from the graph determine approximately the required values of x.

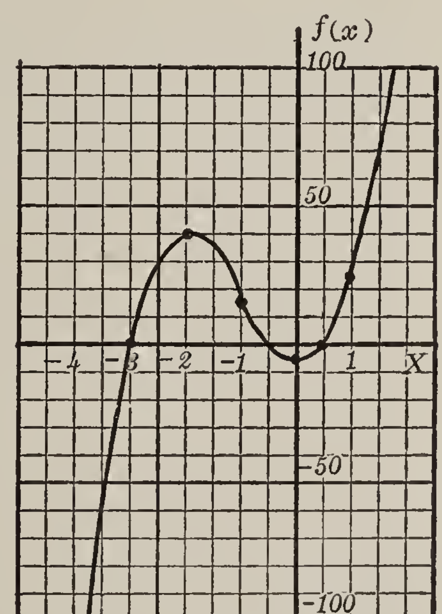


FIG. 7

2. $x^3-2x^2-7x-4=0$

Synthetic Division. Remainder Theorem

16. The work of evaluating functions for different values of x is greatly simplified by means of a process called *synthetic division* and by the use of a theorem called the *remainder theorem.*

17. Synthetic division. The process of *synthetic* division is illustrated in the following example:

Divide $2x^3-7x^2-3x+5$ by $x-2$.

The process of *long* division is as follows:

$$
\begin{array}{l|l}
2x^3-7x^2-3x+5 & x-2 \\ \cline{2-2}
\underline{2x^3-4x^2\quad} & 2x^2-3x-9 \\
\quad -3x^2-3x & \\
\quad \underline{-3x^2+6x\quad} & \\
\qquad\quad -9x+5 & \\
\qquad\quad \underline{-9x+18} & \\
\qquad\qquad -13 &
\end{array}
$$

To shorten the work of this division, omit the various powers of x, writing only the coefficients. The work is now arranged as follows:

$$
\begin{array}{l|l}
2-7-3+5 & 1-2 \\ \cline{2-2}
\underline{2-4\quad} & 2-3-9 \\
\ -3-3 & \\
\ \underline{-3+6\quad} & \\
\qquad -9+5 & \\
\qquad \underline{-9+18} & \\
\qquad\quad -13 &
\end{array}
$$

Next, omit bringing down the terms of the dividend as parts of the remainders. This gives

$$
\begin{array}{rrrr|l}
2-7 & -3 & +5 & & \underline{1-2} \\
\underline{2-4} & & & & 2-3-9 \\
-3 & & & & \\
\underline{-3} & \underline{+6} & & & \\
 & -9 & & & \\
 & \underline{-9} & \underline{+18} & & \\
 & & -13 & &
\end{array}
$$

The coefficients of the quotient may be omitted, as they are equal, respectively, to the first coefficients of the dividend and of the remainders. Similarly, the first coefficients of the partial products may be omitted. Finally, since the first coefficient of the divisor is always 1, it need not be written. This reduces the process to the following:

$$
\begin{array}{rrrr|l}
2-7 & -3 & +5 & & \underline{-2} \\
\underline{-4} & & & & \\
-3 & & & & \\
 & \underline{+6} & & & \\
 & -9 & & & \\
 & & \underline{+18} & & \\
 & & -13 & &
\end{array}
$$

In the process of division, as given above, the partial products -4, $+6$, and $+18$ are subtracted. By changing the -2 in the divisor to $+2$, and thus changing the signs in the partial products, the subtraction of

the partial product is changed to addition. This gives products,

$$\begin{array}{rrrr|l}
2 & -7 & -3 & +5 & \underline{\;2\;} \\
 & 4 & & & \\ \cline{1-2}
 & -3 & & & \\
 & & -6 & & \\ \cline{2-3}
 & & -9 & & \\
 & & & -18 & \\ \cline{3-4}
 & & & -13 &
\end{array}$$

Finally, the work may be condensed into the following form:

$$\begin{array}{rrrr|l}
2 & -7 & -3 & +5 & \underline{\;2\;} \\
 & 4 & -6 & -18 & \\ \cline{1-4}
2 & -3 & -9 & -13 &
\end{array}$$

Notice that the first three successive terms in the lowest line are the *coefficients of the quotient* and the *last* term is the *remainder*. Division in this abbreviated form is called **synthetic division.**

18. Rule for synthetic division. To divide $f(x)$ by $x-a$, *$f(x)$ is arranged in descending powers of x, supplying zeros as coefficients of missing terms.*

The coefficients are written horizontally and the first coefficient is brought down.

This coefficient is multiplied by a and the product added to the second coefficient.

This process is repeated until a product has been added to the last coefficient.

The last sum is the remainder. The preceding sums are the coefficients of x in the quotient, arranged in descending order.

EXERCISES

Divide synthetically

1. x^4-3x^3+4x+2 by $x-3$

2. $3x^3-4x+7$ by $x-1$

3. $5x^3+2x^2-3$ by $x-5$

4. $4x^3+x^2-3x-1$ by $x+2$

Change $x+2$ to $x-(-2)$.

5. $2x^4+6x-5$ by $x+1$

19. Remainder theorem. The following exercises show that the *value* of $f(x)$ for $x=a$ may be found by *dividing* $f(x)$ by $x-a$:

EXERCISES

1. Divide $f(x)=x^2-6x+3$ by $x-2$.

By synthetic division the quotient and remainder are obtained as follows:

$$\begin{array}{r|l} 1-6+3 & \underline{2} \\ 2-8 & \\ \hline 1-4-5 & \end{array}$$

Hence the *quotient* is $x-4$ and the *remainder* -5.

2. Find the value of $f(x)=x^2-6x+3$ for $x=2$.

By substitution, $f(2)=2^2-6\cdot 2+3=-5$.

Thus the *value* of $f(x)$ for $x=2$ and the *remainder* obtained by dividing $f(x)$ by $x-2$ are the same.

If $f(x)$ denotes a function of x in the form

$$ax^n+bx^{n-1}+cx^{n-2}\ \ldots\ .$$

the result of exercises 1 and 2 may be stated as a theorem as follows:

When $f(x)$ is divided by $x-a$, the remainder is $f(a)$.

This principle is called the **remainder theorem.**

20. Proof of the remainder theorem. Since dividend $\equiv$ divisor $\times$ quotient $+$ remainder, it follows that $f(x) \equiv (x-a)Q(x)+R$, where the function $Q(x)$ is the quotient and the constant R the remainder.

Substituting a for x, $f(a) \equiv (a-a)Q(a)+R$.

$\therefore f(a) \equiv 0 \cdot Q(a)+R$.

$\therefore f(a) \equiv R$, which was to be proved.

21. Evaluation of $f(x)$. According to the remainder theorem the *value* of $f(x)$, for $x=a$, may be found by *dividing* $f(x)$ by $x-a$.

EXERCISES

Find the values of the following functions, letting x take all integral values from -4 to $+3$:

1. $10x^3+29x^2-5x-6$

Dividing synthetically by -4,

$$\begin{array}{rrrr|l} 10 & +29 & -5 & -6 & \underline{-4} \\ & -40 & +44 & -156 & \\ \hline 10 & -11 & +39 & \underline{-162} & \end{array}$$

$\therefore f(-4) = -162$

Similarly, find $f(-3)$, $f(-2)$, etc.

2. x^3-2x^2-7x-4

3. $3x^3-4x+7$

Solve the following equations graphically:

4. $x^3-4x^2-2x+8=0$

5. $x^3-3x^2-x+3=0$

6. $x^3+x^2-10x-10=0$

7. $x^3+3x^2+2x=0$

Equations of Degree Higher than the Second Solved by Factoring

22. Factor theorem. In § 19 it was shown that the remainder obtained by dividing $f(x)$ by $x-a$ is $f(a)$. Hence, if the remainder, $f(a)$, is *zero*, the division of $f(x)$ by $x-a$ is *exact* and $x-a$ is a factor of $f(x)$. Thus

$x-a$ is a factor of $f(x)$, if $f(a)=0$.

This principle is called the **factor theorem.**

EXERCISES

The principle in § 22 may be used to factor the following polynomials:

1. $x^3+2x^2-9x-18$

If the polynomial has factors of the form $x-a$, then a must be a divisor of 18. Thus we have the following possibilities for a: ± 1, ± 2, ± 3, ± 6, ± 9, ± 18.

To find $f(1)$, divide synthetically by 1,

$$\begin{array}{rrrr|l} 1 & +2 & -9 & -18 & \underline{1} \\ & 1 & +3 & -\ 6 & \\ \hline 1 & +3 & -6 & -24 & \end{array}$$

Since $f(1)=-24$, it follows that $x-1$ is not a factor. Why not?

Similarly, we find

$f(-1)=-8 \therefore x+1$ is not a factor.

$f(2)=-20 \therefore x-2$ is not a factor.

$f(-2)=\ \ 0 \therefore x+2$ is a factor. Why?

Show that the other factor is x^2-9, which may be factored as the difference of two squares.

2. $x^3-6x^2+11x-6$

Since there are no powers of x missing and since the signs are alternately $+$ and $-$, $f(x)$ cannot be 0 for negative values of x. Why?

Hence the only values to be tried are 1, 2, 3, and 6.

3. $x^3+6x^2+11x+6$

Show that $f(x)\neq 0$ for positive values of x.

4. $y^4-5y^3+5y^2+5y-6$

5. $x^3+3x^2y-4y^3$

6. Show that $x-y$ is a factor of x^n-y^n if n is an integer, and find the factors of x^5-y^5.

7. Show that $x+y$ is a factor of x^n-y^n if n is an *even* integer, and factor x^8-y^8.

8. Show that $x+y$ is a factor of x^n+y^n if n is an *odd* integer, and factor x^5+y^5.

9. Show that for *even* values of n the function x^n+y^n has *no* factors of the form $x+y$ or $x-y$.

Solve the following equations:

10. $x^3-7x+6=0$

The factors are $(x-1)$, $(x-2)$, and $(x+3)$.

$\therefore\ (x-1)(x-2)(x+3)=0.$

$$\therefore \quad \left.\begin{cases} x-1=0 \\ x-2=0 \\ x+3=0 \end{cases}\right\} \text{ satisfy the given equation.}$$

$$\text{Hence} \quad \begin{cases} x_1=1\dagger \\ x_2=2 \\ x_3=-3. \end{cases}$$

11. $y^3-19y-30=0$

12. $x^3-5x-2=0$

13. $t^3-3t+2=0$

14. $2y^3-y^2-5y-2=0$

15. $y^2+\frac{1}{y^2}+y+\frac{1}{y}=4$

16. $x^3-4x-8x^2+32=0$

17. $y^3-y^2=6y$

18. $x^3-1=0$

Notice that the roots of this equation are the three cube roots of 1.

23. Equations of degree higher than the second solved like quadratics.

EXERCISES

Solve the following equations:

1. $(x+2)^2+3(x+2)=18$

By factoring, we have

$$(x+2+6)(x+2-3)=0$$

$$\therefore \begin{cases} x_1=-8 \\ x_2=1 \end{cases}$$

2. $y^6+2y^3-80=0$

3. $(5y-4)^2-2(5y-4)-63=0$

† A letter with a subscript, as x_1, is read "x sub one", or "x one."

4. $(x^2-6x)^2+5(x^2-6x+20)-136=0$

Change the equation to the form

$$(x^2-6x)^2+5(x^2-6x)+100-136=0,$$

or

$$(x^2-6x)^2+5(x^2-6x)-36=0$$

5. $3(y^2+3y+1)^2-7(y^2+3y+1)+4=0$

6. $y^2+y-2-\dfrac{3}{y^2+y}=0$

7. $2(y^2-3)+\dfrac{8}{y^2-3}+17=0$

The Function $\dfrac{c}{x}$

24. Inverse variation. The equivalent statements y *varies inversely as* x and y *is inversely proportional to* x are expressed algebraically in the form $y=\dfrac{c}{x}$. Thus the statement, the force of gravity varies inversely as the square of the distance, is written $f=\dfrac{c}{d^2}$. In this equation c is the **constant of variation** and f is a *function* of d.

EXERCISES

Express the following statements by means of equations:

1. The number of vibrations a pendulum makes in one second varies inversely as the square root of the length.

2. The amount of heat received from a stove varies inversely as the square of the distance from it.

3. The volume of gas inclosed in a cylinder varies inversely as the pressure.

4. The pressure which a given quantity of air at constant temperature exerts against the walls of a containing vessel is inversely proportional to the volume occupied (Boyle's law).

Solve the following problems:

5. The intensity of light on an object varies inversely as the square of the distance from the source of light to the object. A screen 15 ft. from a lamp is moved to a distance of 5 ft. from it. How much does this increase the intensity?

6. A pendulum 39.1 in. long makes one vibration in a second. Find the length of a pendulum vibrating 4 times a second.

Use exercise 1.

7. The volume of a gas confined in a cylinder is 1 cu. ft., when the pressure is 5 pounds. What is the volume when the pressure is 20 pounds?

Use exercise 4.

25. Graph of $\frac{c}{x}$. Corresponding values of x and $f(x)$ for $c=1$ are given in the table, Fig. 8.

x	$f(x)$
4	$\frac{1}{4}$
3	$\frac{1}{3}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2
$\frac{1}{4}$	4
$\frac{1}{5}$	5
$\frac{1}{6}$	6
.	.
.	.
.	.
.	.
$-\frac{1}{6}$	-6
$-\frac{1}{5}$	-5
$-\frac{1}{4}$	-4
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-3	$-\frac{1}{3}$
-4	$-\frac{1}{4}$

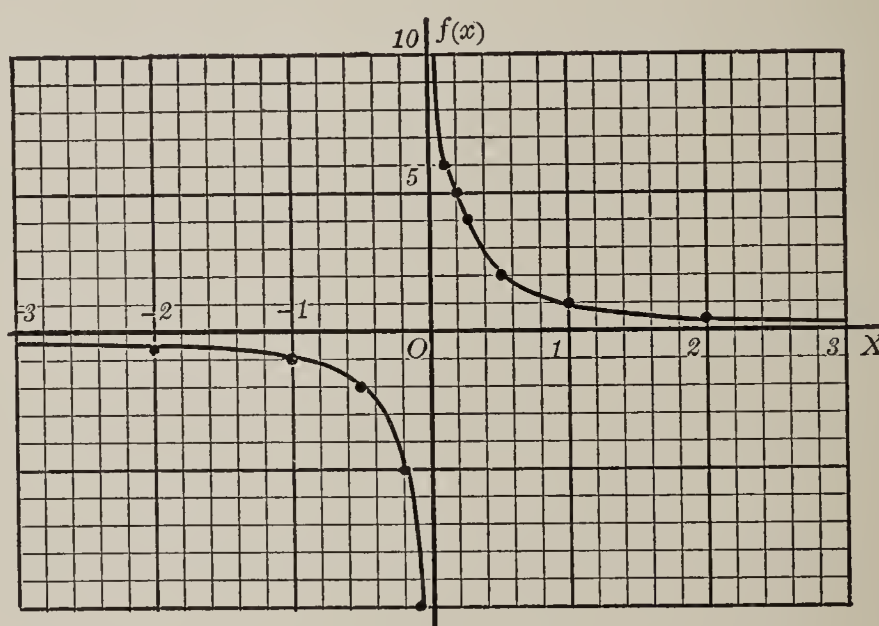


Fig. 8

Plot the pairs of values given in the table, Fig. 8. The curve obtained consists of two branches which do not touch either axis. Since both branches are obtained

from the same equation, $f(x)=\frac{1}{x}$, they are said to form one and the same curve. This curve is called a **hyperbola.**

Show that $\frac{1}{x}$ decreases, if x increases.

By taking x large enough, $\frac{1}{x}$ can be made *smaller* than any number whatsoever.

If x is positive and decreases, $\frac{1}{x}$ increases. By taking x small enough, $\frac{1}{x}$ can be made *larger* than any number however great. Often this fact is expressed briefly by the statement $\frac{1}{0}=\infty$. However, this does not mean that 1 divided by 0 has a value. It means that $\frac{1}{x}$ *increases without bound as x approaches zero as a limit.*

EXERCISES

1. Graph the function $f(x)=\frac{12}{x}$.

2. Graph the function $f(x)=\frac{8}{x}$.

3. By means of the graph of the function $\frac{c}{x}$ find a meaning for the expression $\frac{c}{0}$.

4. Discuss the changes of $\frac{c}{x}$ as x changes from $-\infty$ to $+\infty$.

26. Joint variation. The area of a triangle varies as the product of the base by the altitude. It is said to vary *jointly* as the base and altitude.

If a train moves with a uniform speed the distance varies jointly as the rate and time. This may be expressed algebraically by means of the equation $d=rt$, d denoting the distance, r the rate, and t the time.

The statement **y varies jointly as x and z** is expressed in symbols by means of the equation $y=cxz$.

EXERCISES

Express by an equation each of the following statements:

1. The distance passed over by a train varies jointly as the rate and time.

2. The pressure of water on the bottom of a basin in which it is contained varies jointly as the area of the bottom and the depth of the water.

3. The pressure of wind on a wall varies jointly as the area of the surface and the square of the velocity of the wind.

9. The volume of a cylinder varies jointly as the area of the base and altitude. Compare the volumes of two cylinders whose altitudes are in the ratio 1:2.

27. Direct and inverse variation. If $y=\frac{cx}{z}$ then y **varies directly as x and inversely as z.**

EXERCISES

Express by an equation each of the following statements:

1. The cost of posts for a fence varies directly as the length of the fence and inversely as the distance between the posts.

2. The resistance to an electric current varies directly as the length of the wire and inversely as the cross-section.

3. The current furnished by different galvanic cells is directly proportional to the electromotive force and inversely proportional to the resistance of the circuit (Ohm's law).

Solve the following problems:

4. If y varies directly as x and inversely as z, and if $y=14$ when $x=7$ and $z=1$, find y when $x=84$ and $z=6$.

5. The interest on a sum varies jointly as the rate and principal. If in a certain number of years the interest on $2,000 at 5 per cent is $400, what is the interest on a principal of $2,500 at $5\frac{1}{2}$ per cent in the same time?

Summary

28. The following exercises summarize the terms, symbols, and processes taught in this chapter:

1. Give the meaning of the following terms:

function	intercept
variable	direct variation
constant	inverse variation
evaluation of a function	joint variation
linear, quadratic, cubic, function	direct and inverse variation
zero of a function	parabola
abscissa, ordinate	hyperbola
	synthetic division

2. Explain the meaning of the following symbols: $f(x)$, $F(x)$, $-\infty$, $+\infty$.

3. Tell how to make the graph of $f(x)$, if $f(x)$ is linear; quadratic; cubic.

4. Explain the use of synthetic division in evaluating a function of x.

5. Explain how to solve equations in one unknown of the second degree, or higher,

1. By means of the graph
2. By factoring

6. Show that a straight line can be represented by an equation of the form $f(x)=mx+b$.

7. State and prove the remainder theorem.

8. State the factor theorem.

9. Represent graphically, $y=x^2-2x$. (Wisconsin.*)

10. The attraction of gravitation at points outside the earth's surface varies inversely as the square of the distance from the

* (Wisconsin) means: taken from an entrance examination given by the University of Wisconsin.

earth's center. If the attraction on a certain body is 9 lb. at the surface of the earth, at what altitude above the surface would the attraction on the same body be reduced to 4 pounds? (Take the radius of the earth as 4,000 miles.) (Harvard.)

11. Lights of equal brightness are placed in three corners of a square room. Show that the intensity of the illumination at the fourth corner is $\frac{5}{12}$ that at the center of the room. Given that the intensity of the illumination varies inversely as the square of the distance from the light. (Harvard.)

CHAPTER II

TRIGONOMETRIC FUNCTIONS

29. Angles in general. In the preceding course* the sine, cosine, and tangent of *acute* angles were defined. These functions have been used in the solution of *right* triangles, § 257, *S.-Y.M.* However, in the *general* triangle, *obtuse* angles as well as acute angles are found. To work with the general triangle the notion of the trigonometric functions of an angle must be extended to include angles that are greater than 90°.

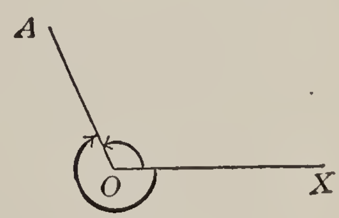


Fig. 9

30. Angle as amount of rotation. The angle XOA, Fig. 9, is considered as generated by the rotation of a line from OX around to the position OA. The line OX is called the **initial side** and OA the **terminal side** of the angle. If the line revolves from OX in the *counter-clockwise* direction, the angle XOA is **positive**; if it revolves in the *clockwise* direction, the angle formed is **negative.**

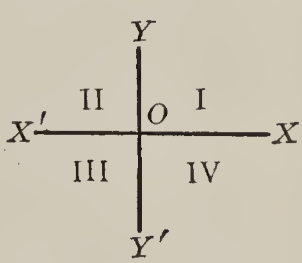


Fig. 10

31. Quadrants. Two lines at right angles, Fig. 10, divide the plane around the point of intersection, O, into four equal parts called **quadrants.** The quadrants are numbered as follows: XOY is the **first** quadrant, YOX' the **second,** $X'OY'$ the **third,** and $Y'OX$ the **fourth.**

* *Second-Year Mathematics,* § 248.

An angle is said to be in the first, second, third, or fourth quadrant according as its *terminal side* lies in the first, second, third, or fourth quadrant, the initial side having been placed on OX.

Hence angles between 0° and 90°, 90° and 180°, 180° and 270°, and 270° and 360° are said to be in the first, second, third, and fourth quadrants, respectively (Fig. 11).

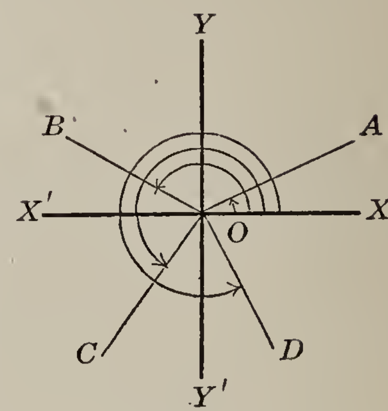


FIG. 11

EXERCISES

1. In which quadrant is $\angle XOA$, Fig. 11? $\angle XOB$? $\angle XOC$? $\angle XOD$?

2. Draw the following angles and in each case state the quadrant in which the angle lies: 20°, 160°, 240°, 315°, 545°, −40°, −220°.

32. Trigonometric functions. Let XOA, Figs. 12 to 15, be a given angle. From any point, P, of the terminal

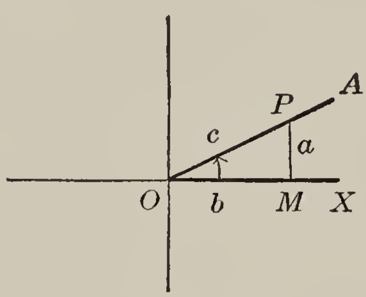


FIG. 12

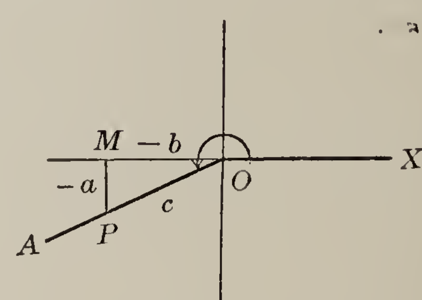


FIG. 14

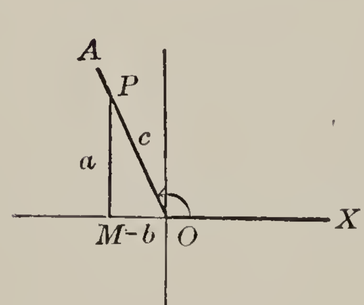


FIG. 13

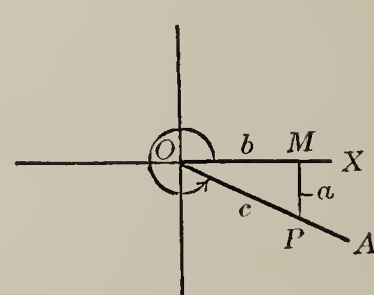


FIG. 15

line OA, drop a perpendicular, PM, to the initial line OX (produced if necessary). This forms a right triangle, MOP.

The **sine** of $\angle XOA$ is the ratio of the side of $\triangle MOP$ that lies opposite the vertex O to the hypotenuse, i.e., $\frac{MP}{OP}$ in each case.

The **cosine** of $\angle XOA$ is the ratio of the side of $\triangle MOP$ that lies adjacent to O to the hypotenuse, i.e., $\frac{OM}{OP}$.

The **tangent** of $\angle XOA$ is the ratio of the side opposite O to the side adjacent, i.e., $\frac{MP}{OM}$.

The **cotangent** of $\angle XOA$ is the ratio of the side adjacent to O to the side opposite O, i.e., $\frac{OM}{MP}$.

The **secant** of $\angle XOA$ is the ratio of the hypotenuse to the side adjacent to O, i.e., $\frac{OP}{OM}$.

The **cosecant** of $\angle XOA$ is the ratio of the hypotenuse to the side opposite O, i.e., $\frac{OP}{MP}$.

These are called the *ratio-definitions* of the trigonometric functions.

Suggest why these ratios are called *functions.*

33. Signs of the functions in each quadrant. If the side opposite to the vertex extends *upward* from the initial line OX it is considered *positive,* if it extends *downward* it is *negative.*

If the *adjacent side* extends to the *right* of O it is *positive,* if to the *left* it is *negative.*

The *hypotenuse* is always regarded as *positive.*

Denoting the measure of angle XOA by the Greek letter α (*alpha*) and the lengths of the sides of $\triangle MOP$ by a, b, and c, respectively, show that the statements of § 32 take the following form:

Functions \ Quadrants	I	II	III	IV
Sine α	$+\frac{a}{c}$	$+\frac{a}{c}$	$-\frac{a}{c}$	$-\frac{a}{c}$
Cosine α	$+\frac{b}{c}$	$-\frac{b}{c}$	$-\frac{b}{c}$	$+\frac{b}{c}$
Tangent α	$+\frac{a}{b}$	$-\frac{a}{b}$	$+\frac{a}{b}$	$-\frac{a}{b}$
Cotangent α	$+\frac{b}{a}$	$-\frac{b}{a}$	$+\frac{b}{a}$	$-\frac{b}{a}$
Secant α	$+\frac{c}{b}$	$-\frac{c}{b}$	$-\frac{c}{b}$	$+\frac{c}{b}$
Cosecant α	$+\frac{c}{a}$	$+\frac{c}{a}$	$-\frac{c}{a}$	$-\frac{c}{a}$

The *signs* of the functions in the various quadrants should be thoroughly well known. The following diagrams,

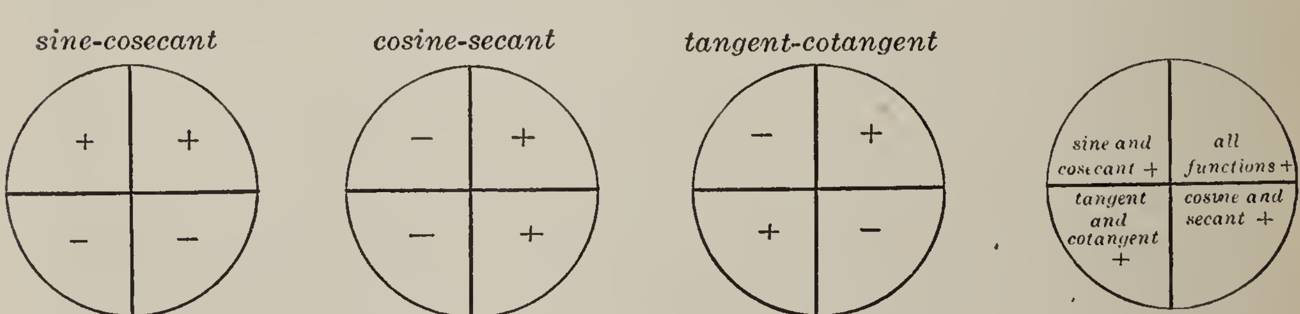


Fig. 16

Fig. 16, will be helpful in remembering the correct algebraic signs of the various functions.

34. Values of the trigonometric functions found by means of a drawing. In the following exercises construct on squared paper the given angle. Draw the defining triangle MOP as in Fig. 17. Measure the sides of the triangle and determine the algebraic signs and the numerical values of the functions of the given angle.

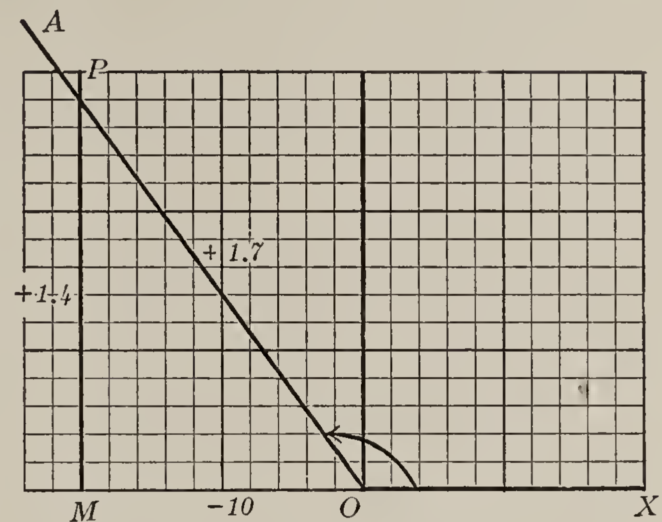


FIG. 17

EXERCISES

1. Find the value of the sine of 125°, Fig. 17.

2. Find the values of the functions of the following angles: 140°, 220°, 245°, 315°.

35. Abbreviations* of the names of the functions. The expressions sine of angle α, cosine of angle α, tangent of angle α, etc., are usually written in the following abbreviated forms: *sin* α, *cos* α, *tan* α, *cot* α, *sec* α, and *csc* α.

36. Given the value of one function of an angle to construct the angle.

EXERCISES

1. Given $\tan \alpha = \frac{3}{4}$. Construct angle α.

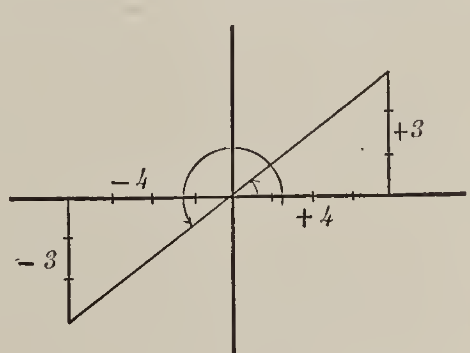


FIG. 18

Show that there are two angles whose tangent is $\frac{3}{4}$, one in the first and one in the third quadrant, Fig. 18. Construct these angles and measure them with a protractor.

* See note, p. 138, *Second-Year Mathematics*.

2. Construct angles A, x and α, having given $\cos A=\frac{2}{3}$, $\tan x=-3$, and $\cot \alpha=\sqrt{3}$.

3. Construct the angles x and y given by $\tan x=\frac{3}{4}$, $\cos y=\frac{1}{5}$.

4. Construct angles α, having given $\cos \alpha=-\frac{2}{3}$, $\sin \alpha=+\frac{4}{5}$, $\tan \alpha=3$, $\tan \alpha=-3$, $\cot \alpha=4$.

37. Inverse functions. According to § 36 an angle may be found if the value of *one* of the trigonometric functions is known. Thus, the equation $\sin x=\frac{1}{2}$, *determines* x as an angle, or arc,* whose sine is $\frac{1}{2}$. The statement *x is an angle whose sine is y* is usually written briefly: $x=\sin^{-1}y$ or $x=\text{arc}\sin y$, read *inverse sine* and *arc sine* respectively.

Give the meaning of the following:

$$x=\tan^{-1}3,\ y=\text{arc}\cos\left(-\tfrac{1}{2}\right),\ A=\text{arc}\sin\tfrac{3}{4}.$$

38. Given the value of one function of an angle, to determine the values of the other functions.

EXERCISES

1. Given $\tan \alpha=-\frac{4}{3}$, α being in the second quadrant, find the other functions of α.

Construct $\triangle MOP$, Fig. 19, having $OM=-3$, $MP=4$. Compute OP.

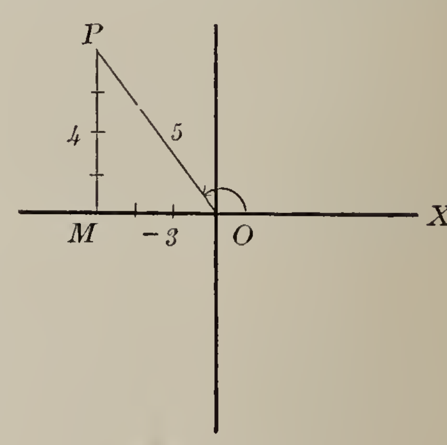


FIG. 19

2. Let $\cot x=\frac{3}{7}$. To find the other functions of x, x being in the third quadrant.

3. Given $\tan x=\frac{3}{4}$. If x is in the third quadrant, give the values of the other functions.

* Since an angle whose vertex is at the center of a circle has the same measure, in degrees, as the intercepted arc, the trigonometric functions may be regarded as functions of the *arc*, instead of the *angle*.

4. Let $\sin a = \frac{18}{25}$, find $\cot a$, Fig. 20.

5. If $\angle A$ is in the third quadrant and $\tan A = \frac{13}{6}$, find $\sec A$ and $\sin A$.

6. Find the values of the functions of x if $\sec x = \frac{3}{2}$.

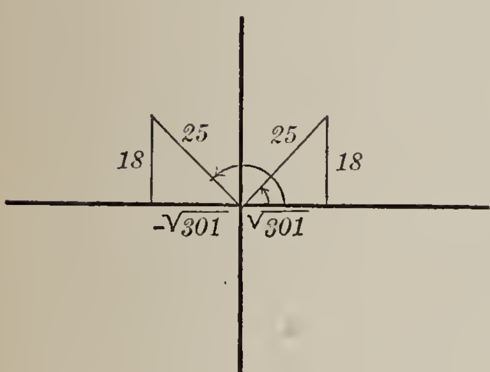


FIG. 20

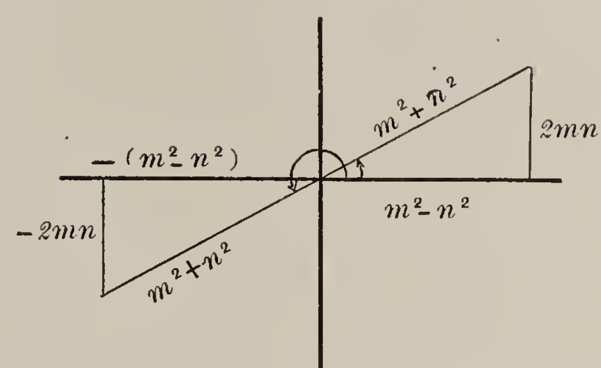


FIG. 21

7. Find $\cos x$, if $\tan x = \dfrac{2mn}{m^2-n^2}$, Fig. 21.

8. If $\sin a = \dfrac{2}{1+a^2}$, find $\cos a$.

9. If $\tan a = \dfrac{2xy}{x^2-y^2}$, find $\sin a$.

10. If $x = \text{arc} \sin y$, find $\tan x$ in terms of y.

Changes of the Trigonometric Functions as the Angle Changes from 0° to 360°

39. Trigonometric functions represented by lines. Since the trigonometric functions are *ratios* it is possible to represent them graphically by means of line-segments. This simplifies greatly the study of the changes of the functions that depend upon the changes of the angle.

40. Line representation of the sine and cosine functions. Let $\angle XOA$, Fig. 22, be any angle.

With a radius equal to 1 and the center at O draw a circle.

A circle of radius 1 is called a *unit-circle.*

From the point of intersection, P, of the unit-circle and side OA draw $PM \perp OX$.

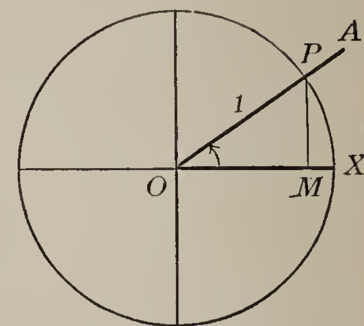


FIG. 22

Denoting $\angle XOA$ by α, we have

$$\sin\ \alpha = \frac{MP}{OP} = \frac{MP}{1} = MP,$$ i.e., the measure of MP is the same as $\sin\ \alpha$. The segment MP represents $\sin\ \alpha$.

Similarly, $\cos\ \alpha = \dfrac{OM}{OP} = \dfrac{OM}{1} = OM$, i.e., the measure of OM is the same as $\cos\ \alpha$. Hence, OM represents $\cos\ \alpha$.

EXERCISE

Draw angles lying in the second, third, and fourth quadrants and represent by line-segments the values of the sine and the cosine functions.

41. Changes of the sine and the cosine functions. In Fig. 23 M_1P_1, M_2P_2, etc., represent $\sin\ \alpha$, and OM_1, OM_2, etc., represent $\cos\ \alpha$.

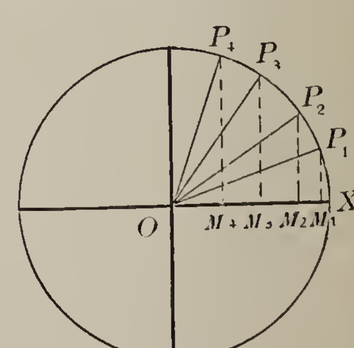


FIG. 23

As α *decreases*, $\sin\ \alpha$ *decreases* and $\cos\ \alpha$ *increases*. When OP coincides with OX, $\alpha = 0$, $\sin 0° = 0$ and $\cos 0° = 1$.

As α *increases* from 0° to 90°, $\sin\ \alpha$ *increases* and $\cos\ \alpha$ *decreases*, both being *positive*.

When $\alpha = 90°$, MP coincides with OP and $\sin 90° = +1$, while $\cos 90° = 0$.

As a *increases* from 90° to 180°, sin a *decreases* from 1 to 0 and cos a *decreases* from 0 to -1, Fig. 24.

As a *increases* from 180° to 270°, sin a *decreases* from 0 to -1 and cos a *increases* from -1 to 0, Fig. 25.

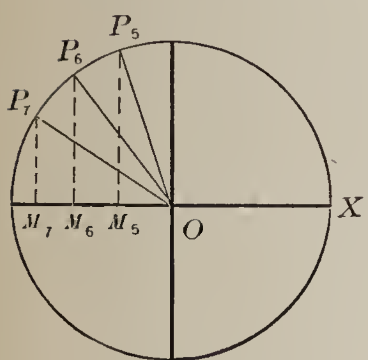


Fig. 24

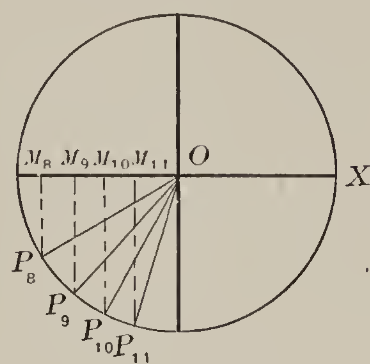


Fig. 25

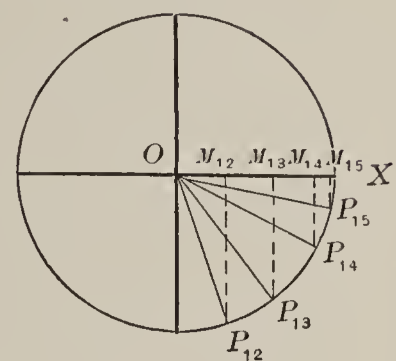


Fig. 26

As a *increases* from 270° to 360°, sin a *increases* from -1 to 0 and cos a *increases* from 0 to 1, Fig. 26.

The following table gives the values of sin a and cos a for special values of a, a being less than, or at most equal to, 360°:

Angle / Function	0°	30°	45°	60°	90°	180°	270°	360°
Sine....................	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cosine.................	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1

The sine and the cosine functions cannot be greater than $+1$ and not less than -1. Why?

EXERCISES

1. Describe the variation of sin x as x increases from 0° to 360°. Illustrate by means of a figure.

2. Describe the variation of *cos* x as x increases from 0° to 360°. Illustrate by means of a figure.

42. Line representation of the tangent and secant functions. Let $\angle XOA$, Fig. 27, be any angle. With radius equal to 1 and with O as center draw a circle. At X draw XT tangent to circle O.

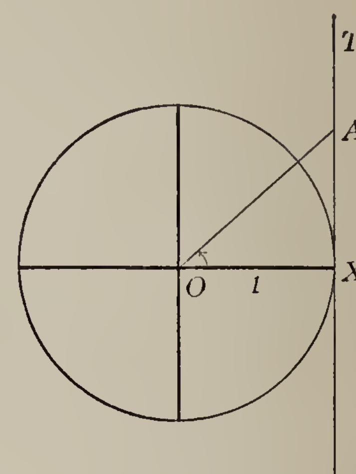


FIG. 27

Denoting $\angle XOA$ by α, we have $\tan \alpha = \frac{XA}{OX} = \frac{XA}{1} = XA$, i.e., the part of the tangent at X intercepted by the initial and terminal sides of $\angle XOA$ has the same measure as $\tan \alpha$. Hence XA represents $\tan \alpha$.

Similarly, $\sec \alpha = \frac{OA}{OX} = \frac{OA}{1} = OA$, i.e., the measure of OA is the same as the value of $\sec \alpha$. Hence, $\sec \alpha$ is represented by OA.

43. Changes of the tangent and secant functions. As α *decreases*, Fig. 28, XA *decreases*. For $\alpha = 0°$, $XA = 0$, i.e., $\tan \alpha = 0$.

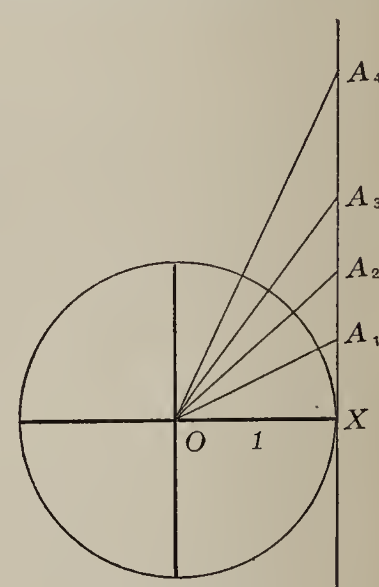


FIG. 28

As α *increases* from 0° to 90°, $\tan \alpha$ *increases*.

As α approaches nearer and nearer to 90°, XA *increases without bound*. Thus, $\tan \alpha$ has no definite value when $\alpha = 90°$. Often this fact is expressed symbolically by the statement $\tan 90° = +\infty$. However, this statement does not mean that 90° has a tangent. It means that *$\tan \alpha$, remaining positive, increases without bound as α approaches* 90° *as a limit.*

Show that $\sec \alpha$ increases from 1 to $+\infty$, as α changes from 0° to 90°.

Show that the sign of $\sec \alpha$, Fig. 28, is the same as the sign of $\cos \alpha$.

When α lies in the second quadrant, Fig. 29, tan α is represented by the part of the tangent at X which is intercepted between the initial side OX and the extension of the terminal side of $\angle XOA$. The fact that XA_1, XA_2, etc., extend *downward* from OX shows that tan α is *negative* in the second quadrant. When α lies in the second quadrant and *decreases* approaching 90° tan α *increases without bound*, always being negative. This is expressed in symbols by means of the statement *tan* $90° = -\infty$. As α approaches 180°, tan α approaches zero.

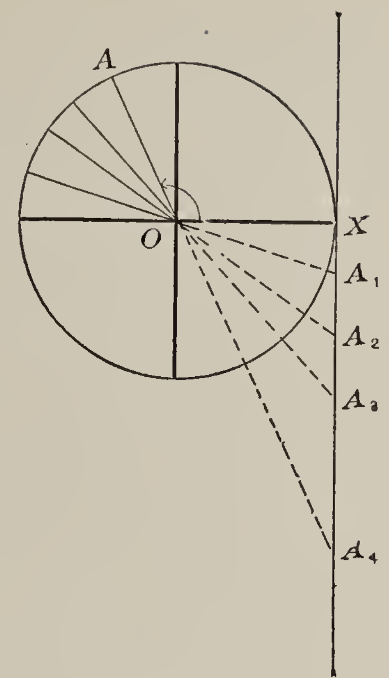


Fig. 29

EXERCISES

1. Show that sec α changes from $-\infty$ to -1 as α changes from 90° to 180°.

2. As α changes from 180° to 270° show that tan α changes from 0 to $+\infty$; that sec α changes from -1 to $-\infty$.

3. As α changes from 270° to 360° show that tan α changes from $-\infty$ to 0; that sec α changes from $+\infty$ to $+1$.

The following table gives the changes of tan α and of sec α, as α changes from 0° to 360°:

Function \ Angle	0°	90°	180°	270°	360°
Tangent........	0	$\pm\infty$	0	$\pm\infty$	0
Secant..........	$+1$	$\pm\infty$	-1	$\mp\infty$	$+1$

44. Line representation of the cotangent and cosecant functions.

Let $\angle XOA$, Fig. 30, be any angle.

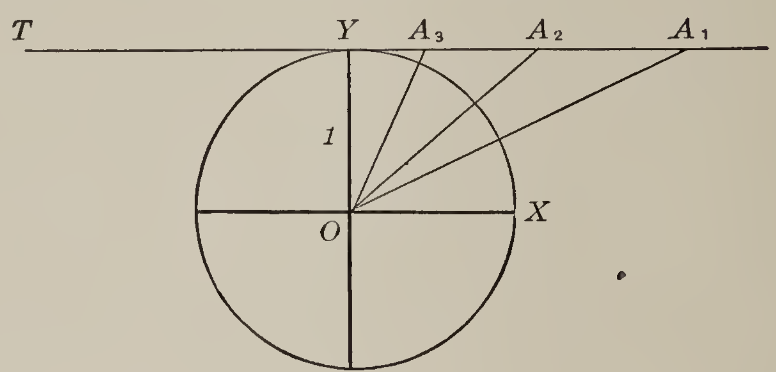


FIG. 30

Draw a unit-circle with the center at O. At Y draw YT tangent to circle O.

Show that $\angle XOA = \angle YAO$.

Denote $\angle XOA$ by α.

Show that $\cot \alpha = \dfrac{YA}{OY} = \dfrac{YA}{1} = YA$, i.e., the part of the tangent at Y which is intercepted by OY and OA has the same numerical measure as $\cot \alpha$.

Similarly, show that $\csc \alpha = \dfrac{OA}{OY} = \dfrac{OA}{1} = OA$.

Hence OA has the same numerical value as $\csc \alpha$ and represents $\csc \alpha$.

When α is *obtuse,* point A is to the left of Y and therefore YA is *negative.*

When α is in the *third* quadrant, show that YA is *positive.*

When α is in the *fourth* quadrant, show that YA is *negative.*

EXERCISE

Describe the variations of $\cot \alpha$ and $\csc \alpha$ as α increases from $0°$ to $360°$. Illustrate by means of figures.

45. Table giving the changes of the functions as the angle changes from 0° to 360°. The following table gives changes of functions of α, as α changes from 0° to 360°:

Function \ Angle	0° to 90°	90° to 180°	180° to 270°	270° to 360°
Sine........	0 to $+1$	$+1$ to 0	0 to -1	-1 to 0
Cosecant.....	$+\infty$ to $+1$	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$
Cosine.......	$+1$ to 0	0 to -1	-1 to 0	0 to $+1$
Secant.......	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$	$+\infty$ to $+1$
Tangent.....	0 to $+\infty$	$-\infty$ to 0	0 to $+\infty$	$-\infty$ to 0
Cotangent....	$+\infty$ to 0	0 to $-\infty$	$+\infty$ to 0	0 to $-\infty$

Graphs of the Trigonometric Functions

46. Radian measure. There are two current methods of measuring angles, viz., *degree measure* and *radian*, or *circular, measure.* The student is already familiar with the first, in which the unit-angle is a degree consisting of $\frac{1}{360}$ of a complete revolution. Before constructing the graphs of the trigonometric functions we will examine the second method and its advantages over the first.

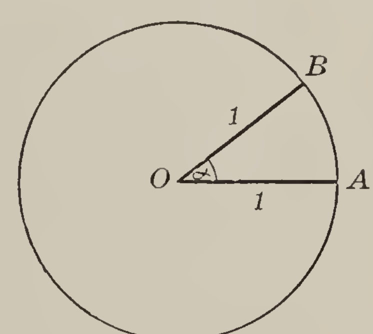


Fig. 31

Let α be the measure, in degrees, of $\angle AOB$, Fig. 31. Draw the unit-circle having the center at the vertex O.

Since α is measured by the intercepted arc, the length of $\widehat{AB}$ may be used to express the value of α. In that case the *unit of measure* is an *arc of unit length*, or the *angle intercepting an arc equal in length to the radius of the circle.* This

method of measuring angles is called *radian measurement*, or *circular measurement*.

47. Radian. The *unit of circular measure*, called the **radian**, is the *angle that intercepts an arc equal in length to the radius of the circle*, Fig. 32. When the unit of measure is not indicated it is understood to be a radian. Thus $\angle ABC = \frac{1}{2}$ means that $\angle ABC$ is $\frac{1}{2}$ of a radian.

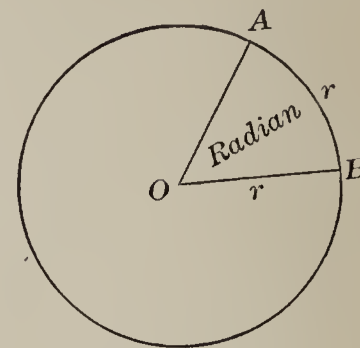


Fig. 32

48. Relation between circular measure and degree measure. Let $\angle BOA$, Fig. 32, be a radian.

Since the length of a semicircumference is πr and since $\overparen{AB}$* $= r$, it follows that

$$\pi \text{ radians} = 180°,$$

where $\pi = 3.14159$ approximately;

$$\therefore \text{a radian} = \left(\frac{180°}{\pi}\right) = 57°.3 \text{ approximately.}$$

Show that $1° = \left(\dfrac{\pi}{180}\right)$ radians.

49. Relation between an angle, the intercepted arc, and the radius of the circle. Let the Greek letter θ (*theta*) denote the number of radians in a given angle, Fig. 33.

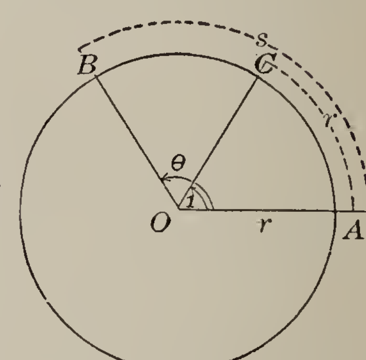


Fig. 33

Then $\dfrac{\theta}{1} = \dfrac{s}{r}$, or $\theta = \dfrac{s}{r}$, or, in words:

The number of radians in an angle is equal to the length of the intercepted arc divided by the radius of the circle.

Show that $s = r\theta$.

Express the equation

$$s = r\theta \text{ in words.}$$

*The symbol $\overparen{AB}$ means arc AB.

EXERCISES

1. Express $10° 15'$ in circular measure.

$$10° 15' = (10\tfrac{1}{4})° = (10\tfrac{1}{4})\left(\frac{\pi}{180}\right) \text{ radians} = .19, \text{ approximately.}$$

2. Express the following angles in circular measure: $10°$, $8° 30'$, $-50°$, $58°$.

3. Find the number of degrees in an angle whose circular measure is $\frac{4}{3}$.

$$\frac{4}{3} \text{ radians} = \frac{4}{3}\left(\frac{180}{\pi}\right)^{\circ} = 76°.39, \text{ approximately.}$$

4. Express in degrees the following angles: $\frac{5}{8}$, $\frac{1}{2}$, $.752$, 3.14.

5. Express in radians the following angles: $0°$, $30°$, $45°$, $60°$, $90°$, $180°$, $270°$, $360°$.

6. Express the following angles in degree measure: $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$, 2π, π, $\frac{\pi}{8}$, $\frac{\pi}{15}$.

7. Give the values of the following functions: $\sin\frac{\pi}{2}$, $\tan\frac{3\pi}{2}$, $\cos\pi$, $\csc\frac{3\pi}{2}$.

8. The radius of a circle is 3 feet. Find the length of the arc intercepted by an angle at the center equal to $1\frac{1}{2}$.

9. The radius of a circle is 10 feet. What is the length of an arc intercepted by an angle of $80°$?

10. What is the circular measure of an angle at the center of a circle intercepting an arc equal to $\frac{2}{3}$ of the radius?

11. Prove that the area of a sector of a circle is $\frac{r^2\theta}{2}$, r being the radius and θ the angle at the center, in radians.

12. Prove that the area of a segment of a circle is $\frac{r^2\theta}{2} - \frac{r^2 \sin\theta}{2}$.

13. The radius of a circle is 10 feet. Find the angle at the center intercepting an arc 2 ft. long. Express the result in degrees and in radians.

50. Graphing the trigonometric functions. To graph the trigonometric functions we may plot the corresponding values of angle and function as obtained from a table of functions. However, the following will show that the line representation affords a very simple way of making the graphs.

51. The sine curve. Lay off on OX, Fig. 34, to a convenient scale, distances representing $a=\frac{\pi}{12}, \frac{\pi}{6} \ldots \frac{\pi}{2} \ldots \pi$, etc., where $\pi=3.14$. At the points thus obtained lay off vertically the corresponding distances representing $\sin a$. Draw a smooth curve through the top points of these vertical lines.

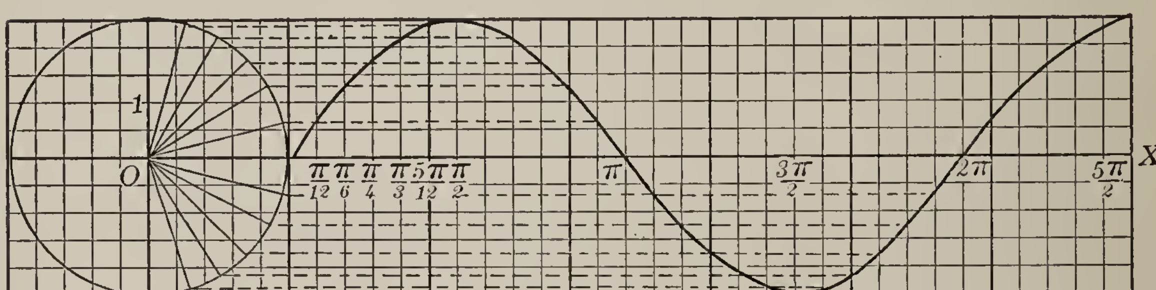


FIG. 34

This is the **sine curve.**

EXERCISES

From a study of Fig. 34 answer the following questions:

1. As a varies from 0 to 360 how does $\sin a$ vary?

2. At what places is the change in $\sin a$ most rapid? Where is the change slowest?

3. How does the curve show that the sine function repeats its values at intervals of 2π, or 360°?

4. What is the largest value of $\sin a$? What is the smallest value?

52. Periodic function. A function whose values are repeated at definite intervals as the variable increases is a **periodic function.**

EXERCISES

1. Show from Fig. 34 that the period of the sine function is 2π, i.e., show that $\sin(\alpha+2\pi)=\sin\alpha$.

2. Show that $\sin(-\alpha)=-\sin\alpha$.

3. Show that $\sin(\pi-\alpha)=\sin\alpha$.

These exercises suggest certain facts about the sine function to be proved in §§ 58–63.

53. The cosine curve. The curve in Fig. 35 is the **cosine curve.** To construct it follow the directions given

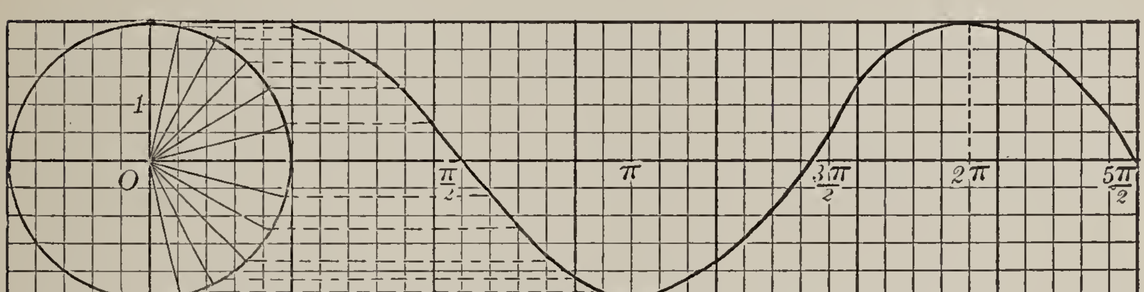


FIG. 35

for the construction of the sine curve, § 51. Notice that the cosine curve has the same shape as the sine curve and differs from it only in position.

EXERCISES

Give a discussion of the cosine curve similar to that given for the sine curve, §§ 51 and 52.

54. The tangent curve. Draw the **tangent curve,** Fig. 36, and give a discussion as in § 51.

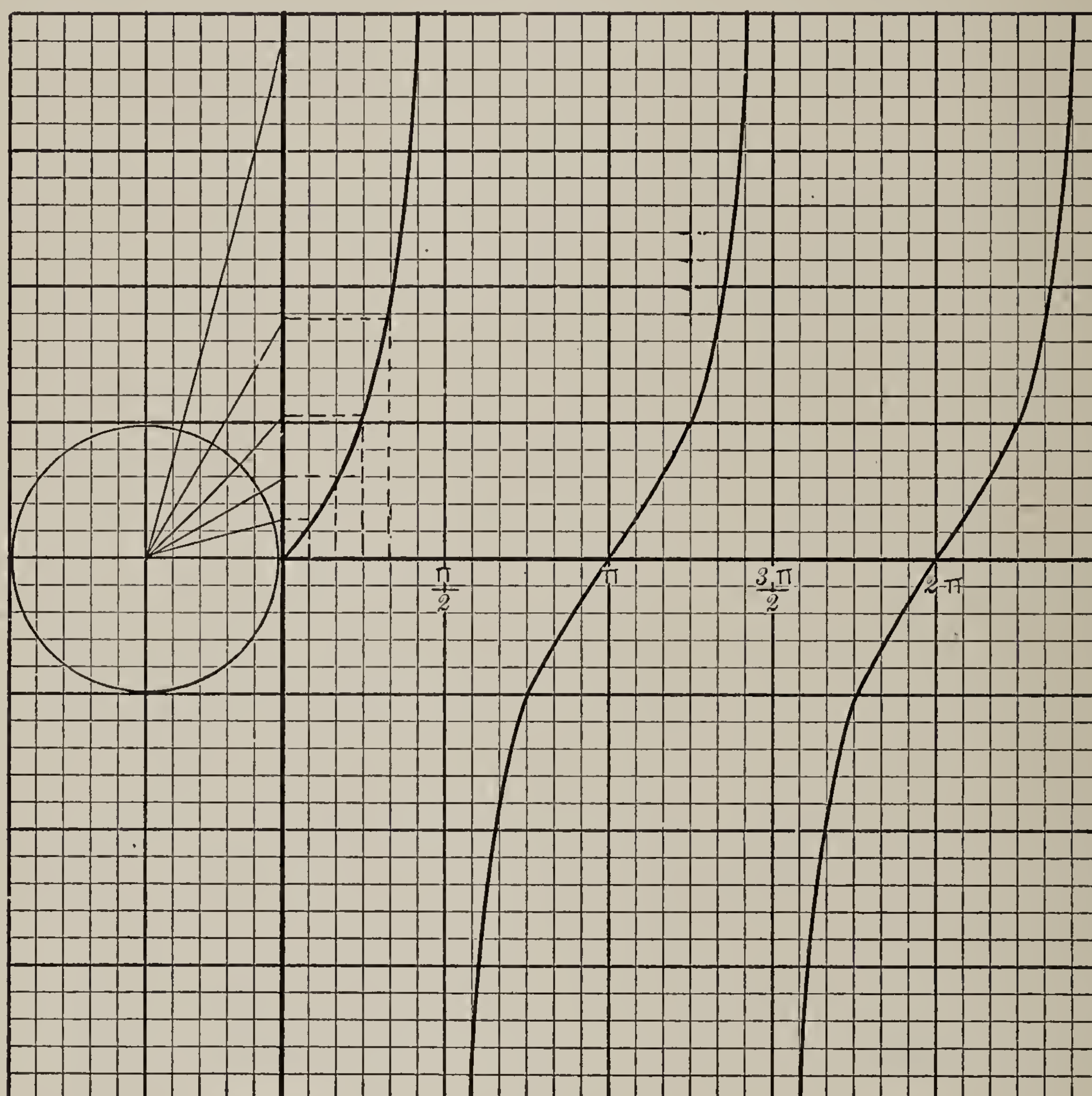


Fig. 36

55. The cotangent curve. Using Fig. 37, draw the cotangent curve, Fig. 38, and give a discussion similar to that given for the sine, cosine, and tangent curves.

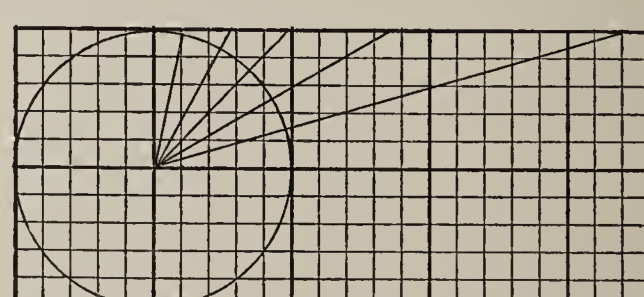


Fig. 37

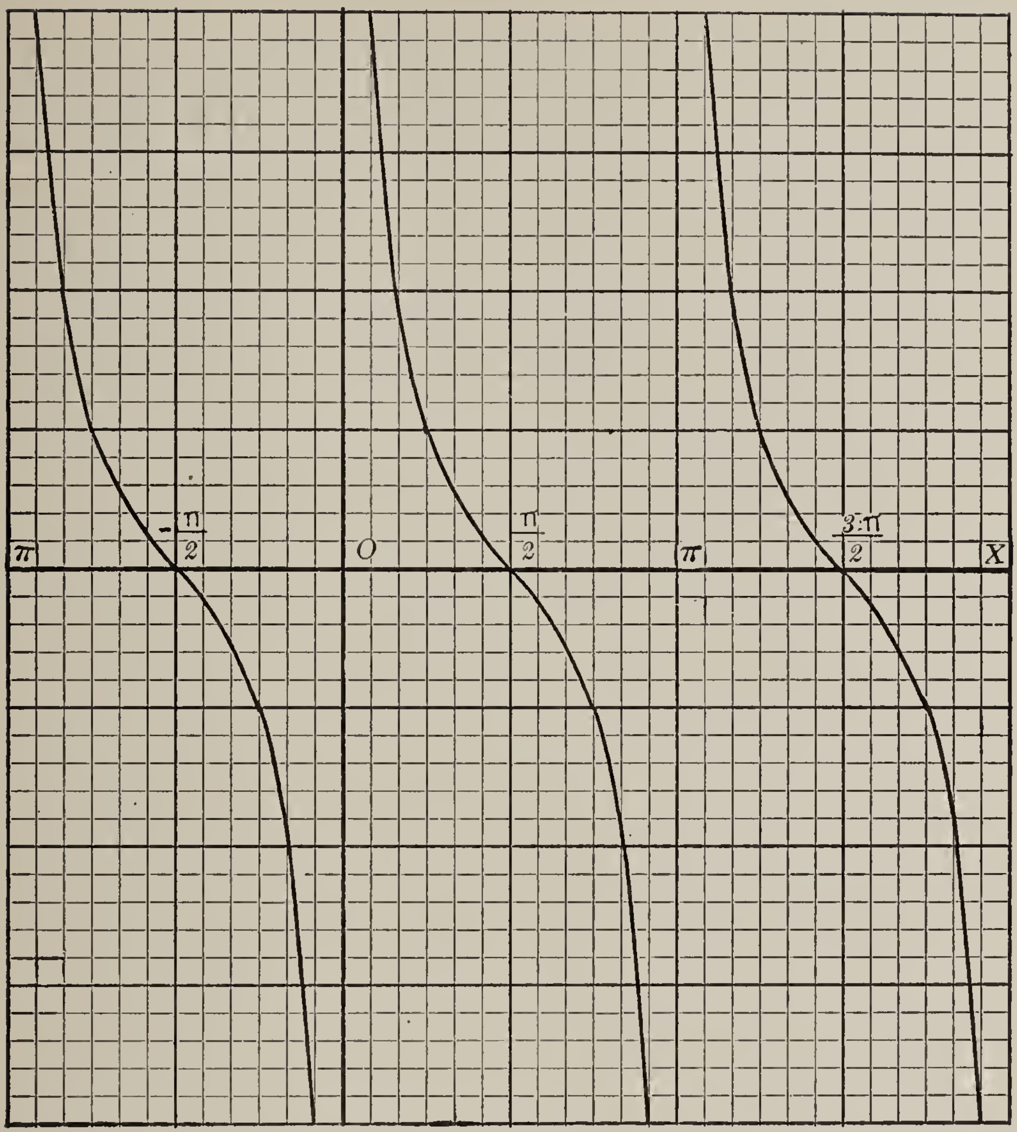

π
$-\frac{\pi}{2}$
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
X

FIG. 38

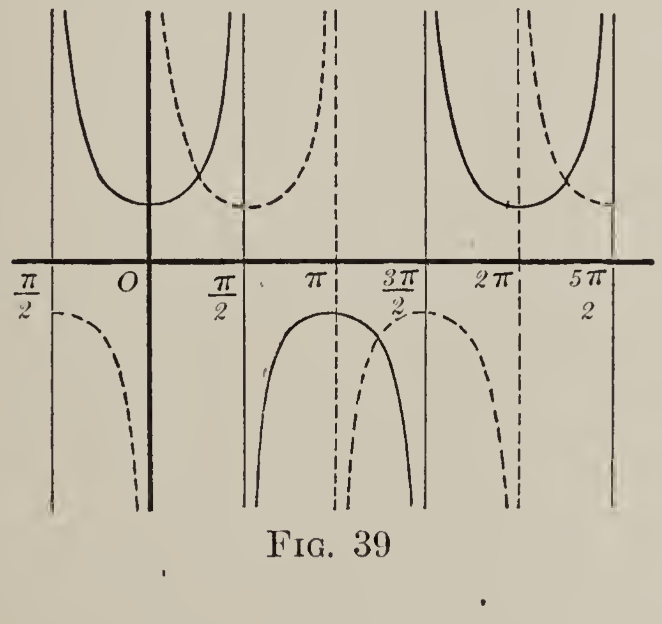

$\frac{\pi}{2}$
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$

FIG. 39

56. The secant and cosecant curves. These curves, given in Fig. 39, may be constructed as in §§ 54 and 55. The solid line represents the secant, the dotted line the cosecant, function.

The Trigonometric Functions of Negative Angles

57. Positive and negative angles. By rotating AB, Fig. 40, around A until it takes the position AC, angle BAC is formed. By rotating AB in the opposite direction, angle BAC' is formed. It is customary to consider an angle **positive** when it is formed by rotating a line *counter-clockwise*, and **negative,** when it is formed by *clockwise* rotation.

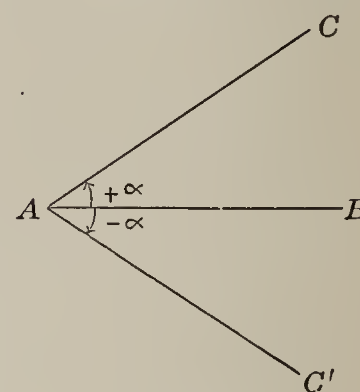


FIG. 40

58. Trigonometric functions of $-\alpha$ in terms of the functions of α. Denoting $\angle BAC$, Figs. 41 to 44, by α, then $\angle BAC' = -\alpha$.

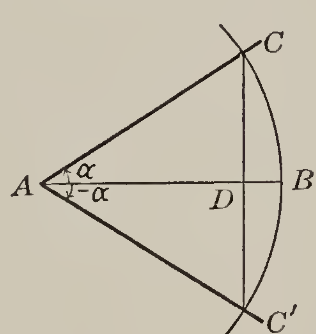


FIG. 41

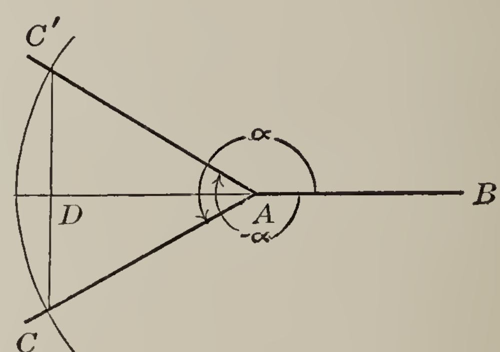


FIG. 43

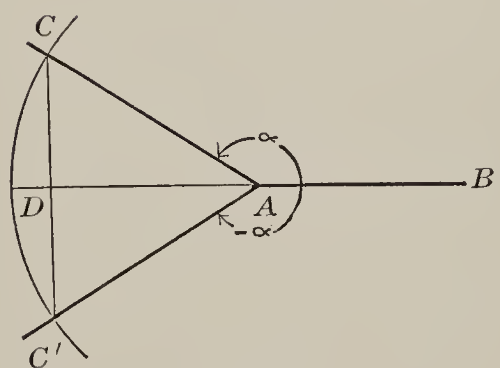


FIG. 42

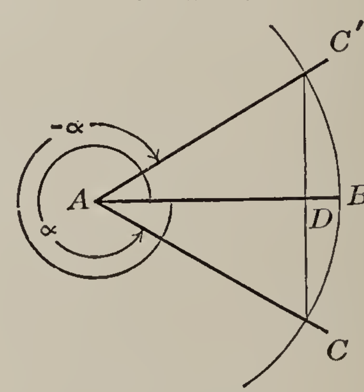


FIG. 44

With A as center and radius equal to 1 draw $\widehat{CC'}$.

Draw chord CC'. Then AD is the perpendicular bisector of $\overline{CC'}$.*

$$\therefore \triangle DAC \cong \triangle DAC'$$

$$\sin \alpha = DC$$

$$\sin(-\alpha) = DC' = -DC = -\sin \alpha$$

Similarly, $\cos(-\alpha) = AD = \cos \alpha$

$$\therefore \frac{\sin(-\alpha)}{\cos(-\alpha)} = \frac{-\sin \alpha}{\cos \alpha}$$

$$\therefore \tan(-\alpha) = -\tan \alpha \qquad \text{Why?}$$

EXERCISES

1. Show that

1. $\cot(-\alpha) = -\cot \alpha$
2. $\sec(-\alpha) = \sec \alpha$
3. $\csc(-\alpha) = \csc \alpha$

2. Express in terms of functions of positive angles, the values of all functions of the following angles:

$$-30°, \ -45°, \ -60°.$$

The exercises in § 58 show that *any trigonometric function of* $-\alpha$ *is equal numerically to the same function of* α, *but differs in sign, with the exception of the cosine and secant.*

59. In § 34 we have seen that the values of the trigonometric functions of *any* angle may be found from a drawing. The values of the functions of angles *less than 90°* may be found by means of special tables. The values of the functions of angles *greater than 90°* can be found by means of certain relations which enable us to express the functions of *any* angle in terms of some function of an angle *less than 90°*. These relations are to be worked out in the following sections.

*$\overline{CC'}$ means chord, or segment CC'.

The Trigonometric Functions of $\left(\frac{\pi}{2} \pm \alpha\right)$ in Terms of Functions of α

60. The relations derived from the graphs of the functions. Let the curves in Fig. 45 represent the sine

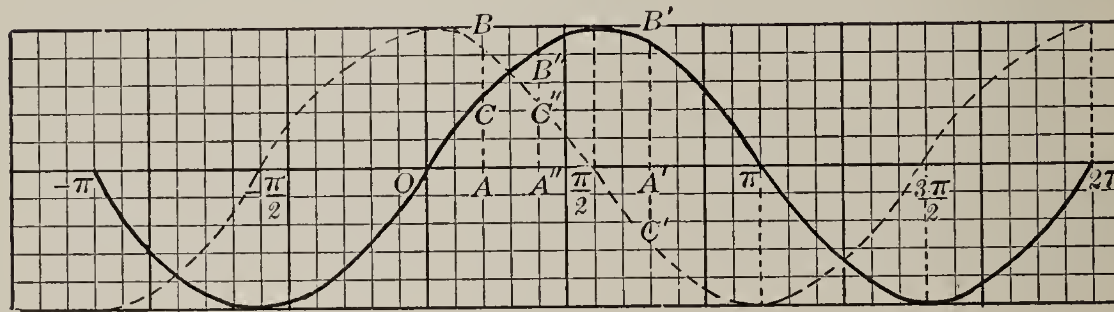


FIG. 45

and cosine functions drawn to the same scale, and let $OA = \alpha$ and $OA' = 90° + \alpha = \frac{\pi}{2} + \alpha$, α being less than 90°. By moving either curve a distance equal to $\frac{\pi}{2}$, it can be made to coincide with the other. Thus AB will coincide with $A'B'$.

Hence, $\sin(90° + \alpha) = \cos\alpha$. (1)

Similarly, the *sine curve* can be made to coincide with the *cosine curve* by moving it to the right a distance equal to $\frac{\pi}{2}$, and then rotating it about OX as an axis.

AC will then coincide with $A'C'$

Hence, $\cos(90° + \alpha) = -\sin\alpha$ (2)

$\therefore \tan(90° + \alpha) = -\cot\alpha$ Why? (3)

and $\cot(90° + \alpha) = -\tan\alpha$ Why? (4)

Show that $\sec(90° + \alpha) = -\csc\alpha$ (5)

and $\csc(90° + \alpha) = \sec\alpha$ (6)

Since $A'B' = A''B''$, it follows that

$\sin(90° - \alpha) = \sin(90° + \alpha) = \cos\alpha$ (7)

Since $A''C'' = A'C'$, it follows that

$$\cos (90° - \alpha) = -\cos (90° + \alpha) = +\sin \alpha \qquad (8)$$

$$\therefore \tan (90° - \alpha) = \cot \alpha \qquad \text{Why?} \qquad (9)$$

and $$\cot (90° - \alpha) = \tan \alpha \qquad \text{Why?} \qquad (10)$$

Show that $\sec (90° - \alpha) = \csc \alpha$ (11)

and $$\csc (90° - \alpha) = \sec \alpha \qquad (12)$$

Sketch roughly the graphs of the tangent and cotangent functions and verify relations (3) and (4).

The relations (1) to (12) may be summarized as follows: *any trigonometric functions of* $(90° - \alpha)$ *is equal to the cofunction of* α; *and any trigonometric function of* $(90° + \alpha)$ *is equal numerically to the cofunction of* α, *but differs in sign with the exception of the sine and cosecant functions.*

61. The relations found in § 60 may be proved as follows:

Denoting $\angle XAB$, Fig. 46, by α, then $\angle ABC = \frac{\pi}{2} - \alpha$.

$$\therefore \frac{a}{c} = \sin \alpha = \cos \left(\frac{\pi}{2} - \alpha\right)$$

$$\frac{b}{c} = \cos \alpha = \sin \left(\frac{\pi}{2} - \alpha\right)$$

$$\frac{a}{b} = \tan \alpha = \cot \left(\frac{\pi}{2} - \alpha\right)$$

$$\frac{b}{a} = \cot \alpha = \tan \left(\frac{\pi}{2} - \alpha\right)$$

$$\frac{c}{b} = \sec \alpha = \csc \left(\frac{\pi}{2} - \alpha\right)$$

$$\frac{c}{a} = \csc \alpha = \sec \left(\frac{\pi}{2} - \alpha\right)$$

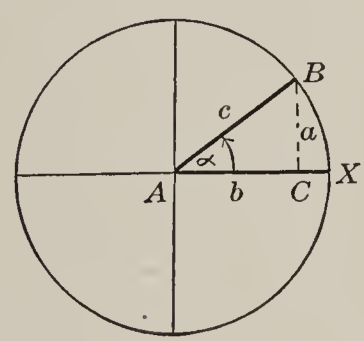


FIG. 46

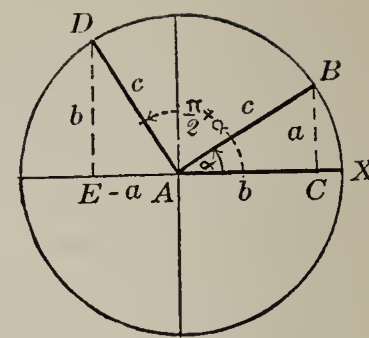


FIG. 47

Let $BAD = \frac{\pi}{2}$, Fig. 47, then $\angle XAD = \left(\frac{\pi}{2} + a\right)$.

Prove that $\triangle CAB \cong \triangle EDA$.

$$\therefore \sin\left(\frac{\pi}{2} + a\right) = \frac{b}{c} = +\cos a.$$

$$\text{and } \cos\left(\frac{\pi}{2} + a\right) = \frac{-a}{c} = -\frac{a}{c} = -\sin a.$$

EXERCISES

1. Prove relations (3) to (6), § 60.

2. Find the exact values of all functions of the following angles: 120°, 135°, 150°.

3. Prove the relations $\sin\left(\frac{\pi}{2} + a\right) = \cos a$ and $\cos\left(\frac{\pi}{2} + a\right) = -\sin a$ for the following cases:

1. When a lies in the second quadrant, Fig. 48.
2. When a lies in the third quadrant, Fig. 49.
3. When a lies in the fourth quadrant, Fig. 50.

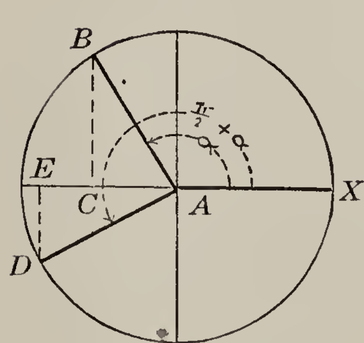


FIG. 48

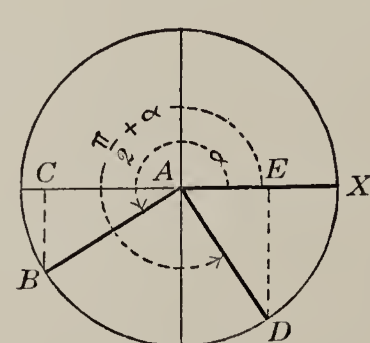


FIG. 49

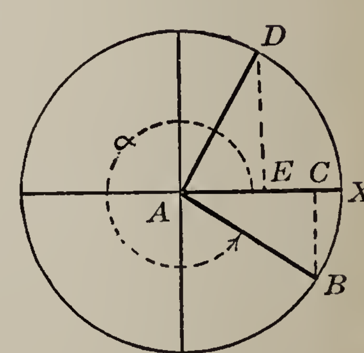


FIG. 50

The Trigonometric Functions of $\left(n \cdot \frac{\pi}{2} \pm \alpha\right)$ in Terms of the Functions of α

62. In the exercises below it is shown that we may express *any* function of $\left(n \cdot \frac{\pi}{2} \pm \alpha\right)$ in terms of functions of α by applying successively the principles of §§ 60 and 61. The results thus obtained are of importance in making trigonometric tables. Since any angle greater than 45° may be changed to the form $(n \cdot 90° \pm \alpha)$, α being less than 45°, *the functions of all angles are expressible as functions of a positive angle less than 45° and may be found from a table giving only the functions of angles from 0° to 45°.*

EXERCISES

Give reasons for the following:

1. 1. $\sin(180° - \alpha) = \sin[90° + (90° - \alpha)] = \cos(90° - \alpha) = \sin\alpha$

2. $\cos(180° - \alpha) = \cos[90° + (90° - \alpha)] = -\sin(90° - \alpha) = -\cos\alpha$

3. $\tan(180° - \alpha) = -\tan\alpha$ 5. $\sec(180° - \alpha) = -\sec\alpha$

4. $\cot(180° - \alpha) = -\cot\alpha$ 6. $\csc(180° - \alpha) = \csc\alpha$

2. 1. $\sin(180° + \alpha) = -\sin\alpha$ 4. $\cot(180° + \alpha) = \cot\alpha$

2. $\cos(180° + \alpha) = -\cos\alpha$ 5. $\sec(180° + \alpha) = -\sec\alpha$

3. $\tan(180° + \alpha) = \tan\alpha$ 6. $\csc(180° + \alpha) = -\csc\alpha$

3. 1. $\sin(270° - \alpha) = \sin[90° + (180° - \alpha)] = \cos(180° - \alpha) = -\cos\alpha$

2. $\cos(270° - \alpha) = -\sin\alpha$ 5. $\sec(270° - \alpha) = -\csc\alpha$

3. $\tan(270° - \alpha) = \cot\alpha$ 6. $\csc(270° - \alpha) = -\sec\alpha$

4. $\cot(270° - \alpha) = \tan\alpha$

4. 1. $\sin(270° + \alpha) = -\cos\alpha$ 4. $\cot(270° + \alpha) = -\tan\alpha$

2. $\cos(270° + \alpha) = \sin\alpha$ 5. $\sec(270° + \alpha) = \csc\alpha$

3. $\tan(270° + \alpha) = -\cot\alpha$ 6. $\csc(270° + \alpha) = -\sec\alpha$

5. 1. $\sin(360° - \alpha) = -\sin\alpha$ 4. $\cot(360° - \alpha) = -\cot\alpha$

2. $\cos(360° - \alpha) = \cos\alpha$ 5. $\sec(360° - \alpha) = \sec\alpha$

3. $\tan(360° - \alpha) = -\tan\alpha$ 6. $\csc(360° - \alpha) = -\csc\alpha$

6. 1. $\sin(360°+a)=\sin a$
2. $\cos(360°+a)=\cos a$
3. $\tan(360°+a)=\tan a$
4. $\cot(360°+a)=\cot a$
5. $\sec(360°+a)=\sec a$
6. $\csc(360°+a)=\csc a$

63. From a study of the preceding exercises we learn the following:

1.* *A function of (an* **even** *multiple of* $90° \pm \alpha$*) is equal numerically to the* **same** *function of* α.

2. *A function of (an* **odd** *multiple of* $90° \pm \alpha$*) is equal numerically to the corresponding* **co***function of* α.

3. *The sign of the result is the same as the sign of the original function in the quadrant in which the angle* $(n \cdot 90° \pm \alpha)$ *lies.*

EXERCISES

Express the following as functions of positive acute angles:

1. $\sin 580°$

$\sin 580° = \sin(6 \cdot 90° + 40°) = -\sin 40°$

2. $\cos 315°$

3. $\sin(-196°)$

4. $\tan(2\frac{7}{12}\pi)$

5. $\cos 120°$

6. $\sin 240°$

7. $\tan(-410°)$

8. $\cos 120°$

9. $\sin 300°$

Express the following as functions of positive angles less than 45°:

10. $\cos(-428°)$

11. $\sin(-84°)$

12. $\csc(834°)$

13. $\tan(-65°)$

14. $\sin 1{,}420°$

15. $\cot 1{,}330°$

* These principles hold even when α is not an acute angle. In case α is greater than 90° the sign is determined by the quadrant in which the angle $\left(n \cdot \frac{\pi}{2} \pm \alpha\right)$ would lie if α were acute.

Find the values of the following:

16. $\cos 180° - \sec^2 45° - 4 \sin 30° + \sqrt{2} \sin 45° + \cos 180° \csc 90°$

17. $\cos \frac{\pi}{3} \cos \frac{2\pi}{3} - \sin \frac{\pi}{3} \sin \frac{2\pi}{3}$

18. $\sin \frac{\pi}{6} \cos \frac{5\pi}{3} + \cos \frac{\pi}{6} \sin \frac{5\pi}{3}$

Simplify—

19. $\tan (\pi - x)$; $\cos \left(\frac{3\pi}{2} - x\right)$; $\sin (270° - x)$; $-\cot (90 + x)$; $\cos (-1{,}230°)$

Summary

64. The student should know the meaning of each of the following terms:

initial side	sine	cosecant
terminal side	cosine	inverse trigonometric functions
quadrant	tangent	circular measure
trigonometric functions	cotangent	radian
	secant	

65. The following exercises review the main topics taught in the chapter:

1. Give the signs of the trigonometric functions in each of the four quadrants.

2. Show how to find the values of the trigonometric functions of given angles by means of a drawing.

3. Show how to construct an angle when the value of one of the trigonometric functions of the angle is known.

4. Explain how the values of the trigonometric functions of an angle may be found when the value of one of the functions is given.

5. Discuss the changes of the trigonometric functions as the angle changes from 0 to 360°, using the straight-line representation.

6. Give the *exact* values of the sine and cosine of the following angles: 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, 330°, 360°.

7. Prove the relations between circular measure and degree measure.

8. Show that the length of an arc of a circle is given by the equation $s=r\theta$, θ being the number of radians in the angle.

9. Draw the graph of each of the trigonometric functions and tell what the graphs show.

10. Express the functions of $(-\alpha)$ in terms of functions of α.

11. Express the functions of $\left(\frac{\pi}{2}\pm\alpha\right)$ in terms of functions of α.

12. Show how to express the functions of *any* angle as functions of a positive angle less than 45°.

CHAPTER III

LINEAR EQUATIONS

Linear Equations in One Unknown

66. Normal form. We have seen that every linear function of x may be changed to the form $ax+b$, § 7. Similarly every linear *equation* in one unknown and many fractional equations may be changed to the form $ax+b=0$. This is called the **normal form** of a *linear* equation in one unknown.

EXERCISES

Change the following equations to the normal form $ax+b=0$:

1. $\dfrac{7x+9}{4}-\left(x-\dfrac{2x-1}{9}\right)=7$

Subtracting as indicated, $\dfrac{7x+9}{4}-x+\dfrac{2x-1}{9}=7.$

Multiplying by 36, $63x+81-36x+8x-4=252.$

Simplifying, we have the normal form, $35x-175=0.$

2. $\dfrac{1}{5}\left(3y-\dfrac{1}{4}\right)+\dfrac{y}{3}-6\left(\dfrac{1}{4}-y\right)=1$

3. $y-\{3y-[4y-(3y-1)]\}=1$

4. $(2-y)(3-y)=(1-y)(5-y)$

‡5. $(2+y)(2y+1)+(2-y)(2y+1)-7y=0$*

6. $(8y+3)^2-3y(13y+9)-(5y+2)^2=0$

7. $\dfrac{13}{z-5}+\dfrac{3}{4}=\dfrac{65}{2z-10}$

8. $ax+b^3=bx+a^3$

‡9. $\dfrac{y}{y+1}-\dfrac{3y}{y+2}+2=0$

10. $\dfrac{a}{x-b}-\dfrac{b}{x-a}=0$

‡11. $(a+b)y=2a-(a-b)y$

12. $(x-a)^2-(x-b)^2=c^2$

13. $(x-a)(x-b)=(x-c)(x-d)$

* All problems and exercises marked ‡ may be omitted.

67. Solution of linear equations. Subtracting b from both sides of the equation $ax+b=0$ and then dividing by a, we have

$$x=\frac{-b}{a}$$

This may be stated in words as follows:

The **root** of a linear equation in *normal form* is obtained by dividing the *negative* of the constant term by the coefficient of the unknown number.

EXERCISES

Solve the following equations:

1. $51\left(\frac{y}{5}-3\right)+.17(y-2)=3.4y$

2. $9.9+\frac{7.2y-5.5}{5}=3.3y-.1x$

3. $2\left(\frac{2a^2-b^2}{a}\right)+b=2y-\frac{b}{a}(3y-4b)$

‡4. $\frac{9}{x-7}-\frac{9}{x-2}=\frac{5}{x-8}-\frac{5}{x+1}$

5. $\frac{5}{y+2}-\frac{2}{4-y^2}=\frac{2}{y-2}$

Change $-\frac{2}{4-y^2}$ to $+\frac{2}{y^2-4}$

6. $\frac{2x-2m}{m-n}-2\frac{(m-n)x-m^2}{m^2-n^2}+\frac{m}{m-n}=0$

‡7. $\frac{3}{z+1}-\frac{z+1}{z-1}=\frac{z^2}{1-z^2}$

8. $\frac{2(y-7)}{y^2+3y-28}+\frac{2-y}{4-y}=\frac{y+3}{y+7}$

‡9. $\frac{3z+2}{z^2+z}+\frac{2(z-5)}{1-z^2}=\frac{z-3}{z^2-z}$

10. $\frac{2y+7}{6y+4}-\frac{15}{8-18y^2}=\frac{3y-5}{9y-6}$

‡11. $\dfrac{2y-1}{2y+1}-\dfrac{8}{1-4y^2}=\dfrac{2y+1}{2y-1}$

12. $\dfrac{5a-3y}{a-3b}-4=\dfrac{y-7a}{3a-b}$

‡13. $\dfrac{m+n}{x}+\dfrac{1}{m+n}=\dfrac{m-n}{x}+\dfrac{1}{m-n}$

‡14. $\dfrac{n^2}{my}-\dfrac{3n-3m}{y}-\dfrac{1}{m}=\dfrac{m^2}{ny}-\dfrac{1}{n}$

Problems Leading to Linear Equations in One Unknown

68. Solve the following problems:

1. Divide 60 into two parts such that the greater part divided by the smaller gives a quotient equal to 2, leaving a remainder equal to 3.

2. What number must be subtracted from each term of the fraction $\frac{7}{13}$ in order that the resulting fraction be equal to $\frac{1}{3}$?

3. The ratio of two numbers is a/b. If the first is increased by m and the second decreased by n, the ratio of the results is $\frac{3}{2}$. What are the numbers?

4. What number must be added to each of the numbers 7, 15, 12, and 24 in order that the resulting sums may form a proportion?

5. A can do a piece of work in 4 days and B can do it in 5 days. How long will it take both together to do the work?

6. A can build a fence in 16 days, B in 18 days, and C in 15 days. How long will it require all working together to build it?

‡**7.** A can do a piece of work in 50 days and B in 30 days. After working together 6 days, A finishes the work alone. How many days does A work alone?

‡**8.** Three men, A, B, and C, can do a piece of work in 30 days. B can do $\frac{2}{3}$ as much as A and $\frac{2}{3}$ as much again as C. Find the time it would take each to do the work alone.

9. Two pipes can fill a tank in 2 and 3 hours respectively. How long would it take to fill the tank if both pipes were open?

10. A train starts for a distant station running at a rate of 30 mi. an hour. Twenty-one minutes later another train starts from the same place, running in the same direction at the rate of 36 mi. an hour. When will the trains meet? When will they be 4 mi. apart?

11. A man travels to a certain place at the rate of 18 mi. an hour. He returns on a road $6\frac{1}{2}$ mi. shorter and makes the trip in 20 min. less time, although traveling only 17 mi. per hour. How long is each road?

12. A man can row in still water at the rate of 8 mi. an hour. It takes him twice as long to go 5 mi. upstream as it does to return. Find the rate of the current.

13. It takes a man 26 hr. to row downstream and back. If he can row $8\frac{2}{3}$ mi. an hour in still water and if the rate of the current is $4\frac{1}{2}$ mi. an hour, how long does it take him to return?

14. A man earned \$90 by working a certain number of days. If he had received 75 cents less a day he would have had to work 10 days more to earn the same amount. Find the number of days in each case.

15. A boy bought a number of pencils for a dollar. Later the price was raised 5 cents per dozen and he received 12 less for a dollar. What was the price per dozen?

16. A certain sum is divided among three persons, such that the first receives \$20 more than the second and the third \$20 less than the second. The whole sum is \$25 more than 4 times as much as the third receives. How much did each receive?

17. Into what two parts must \$5,330 be divided so that the income of one part at 5 per cent shall be twice as much as the income of the other at 4 per cent?

18. The tire of the fore-wheel of a carriage is 9 ft., that of the hind-wheel 12 feet. What distance will the carriage have passed over when the fore-wheels have made 5 more revolutions than the hind-wheels.

19. A man when asked to give his age replied: "Ten years from now I will be twice as old as I was ten years ago." How old was he?

20. A box of oranges was bought at the rate of 15 cents a dozen. Five doz. were given away and the remainder sold at the rate of 2 for 5 cents, leaving a profit of 30 cents on the box. How many were there in the box?

21. A father engaged his son to work 20 days on the following conditions: For each day he worked he was to receive $2, and for each day he was idle he was to forfeit $1. At the end of 20 days he received $34. How many days was he idle?

22. A number of two digits is 2 greater than 17 times the units digit. Find the number if the sum of the digits is 8.

23. The tens digit of a number of two digits is 6. If the order of the digits be reversed, the resulting number is 27 greater than the original number. Find the number.

24. The length of a circle is $2\pi r$, r being the radius and π being 3.14, approximately. The ratio of two circles is 3:4 and one radius exceeds the other by 3. Find the radius of each.

25. The angles of a triangle are in the ratio 2:3:4. How large is each?

26. A basket weighing 56 lb. hangs on a stick 8 ft. long at a point 1 ft. from the middle, while it is being carried by two boys, one at each end. The loads lifted by the boys are in the ratio 5:3. Find how much each boy lifts.

Use the law that the algebraic sum of all leverages is 0, for balance.

27. A steel beam 24 ft. long and weighing 966 lb. is being moved by placing under it an axle borne by a pair of wheels, one end being carried. If the axle is 2 ft. from the middle of the beam what is the weight at the end which is carried.

The weight of the beam may be treated as a load of 966 lb. hanging to the bar at the midpoint.

28. How much water must be added to 12 gallons of a 25 per cent solution of alcohol and water to reduce it to a 10 per cent solution?

29. How much water must be added to a quart of a 20 per cent solution of ammonia to reduce it to a 10 per cent solution?

30. At what time between three and four o'clock are the hands of a clock together?

‡**31.** At what time between eight and nine o'clock are the hands opposite each other?

‡**32.** At what time between five and six o'clock are the hands at right angles?

33. If the radius of a circle is increased by 3 in., the area is increased by 60 square feet. Find the radius of the first circle.

34. By increasing each side of a square by 2 in. the area is increased by 16 square inches. Find the side of the square.

35. If 19 lb. of gold and 10 lb. of silver each lose 1 lb. when weighed in water, how much gold and how much silver is contained in a mass of gold and silver that weighs 80 lb. in air and $72\frac{1}{2}$ lb. in water?

‡**36.** A can do a piece of work in a days and B in b days. How long will it take them to do it working together? Use the result to solve problem 5.

‡**37.** If the radius of a circle is increased by a ft. the area is increased by b square feet. Find the radius of the first circle. Use the result as a formula for solving problem 33.

‡**38.** What number must be added to each of the numbers a, b, c, and d in order that the resulting sums may form a proportion? Use the result as a formula to solve problem 4.

39. Solve the equation $F=\frac{9}{5}C+32$ for C.

40. Solve $s=\frac{n}{2}(a+l)$ for l.

41. Solve $s=\frac{lr-a}{r-1}$ for a.

42. Solve $l(W+w')=l'W'$ for w'.

43. Solve $PV=pv\left(1+\frac{t}{273}\right)$ for t.

44. Solve $\frac{V+v}{V}=\frac{b}{b-p}$ for V.

45. Solve $C=\frac{mE}{R+mr}$ for r.

46. Solve $\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}$ for f_2.

69. Historical note. The theory of linear equations in one unknown is very old. In an ancient Egyptian papyrus, sometimes called the *Reckoning Book of Ahmes*, written about 1700 B.C., there is a chapter of problems whose solutions are commonly referred to as the "Hau-computations." Two samples of these problems are:

I. "Heap, its 2/3, its 1/2, its 1/7, its whole, it makes 33," and

II. "2/3 added to, 1/3 subtracted from, 10 remains."

The word "Hau" (=literally "heap") meant the "unknown," and the mode of thought employed in the solutions was that of the equation. For example, problem I means—

$$\tfrac{2}{3}x+\tfrac{1}{2}x+\tfrac{1}{7}x+x=33,$$

and problem II means—

$$(x+\tfrac{2}{3}x)-\tfrac{1}{3}(x+\tfrac{2}{3}x)=10$$

Though it is known that the Egyptians as early as 1700 B.C. and perhaps as early as 3400 B.C. could solve these problems, all that is known of their method is that it was characterized by their clumsy calculatory processes with unit-fractions.

Whether in the large amount of early mathematical knowledge that passed from Egypt to Greece any of this "Hau-computation" was included is not known. A long interval had to elapse before our type of problem involving linear equations in one unknown acquired a form definite knowledge of which

has survived to our day. This was with Diophantus of Alexandria (third and fourth centuries A.D.). Certain references by earlier writers lead us to think that Diophantus' method was not original, but was only the culmination of methods that had very gradually developed. Diophantus' method is so like ours that we give his rule in his own words:

"When an equation is met which contains the same power of the unknown on both sides with different coefficients, like must be subtracted from like till a term becomes equal to a term. But if upon one or both sides of the equation certain terms are negative, the negative terms must be added on both sides (Operation I) until both sides contain only positive terms, and then like must be subtracted from like (Operation II) until but one term remains on each side of the equation."

Let us illustrate by an example in modern form:

$$8x-11-2x+5=x-4+3x+10$$

$$\text{Operation I}\ \begin{cases} 8x+5+4=x+3x+10+11+2x \\ 8x+9=6x+21 \end{cases}$$

$$\text{Operation II}\ \begin{cases} 8x-6x=21-9 \\ 2x=12 \end{cases}$$

Most of the problems of the book that are devoted to this type lead to the form $ax^m=b$, $(m=1)$.

The Hindus went beyond Diophantus by introducing the idea of negative number, so that with them Diophantus' Operation I was not needed nor used. The Hindus' use of the phrase "subtracting similars" is so like the Diophantine "subtracting like from like" that historians think the Hindus were nevertheless influenced by Greek writings.

The Arabs were the next to work on the theory of equations. They were strongly influenced by Greek mathematics and were never able to rise to the notion of purely negative numbers. Consequently they always prescribed the Operations I and II of Diophantus. The Arabs called one of these operations *aldschebr,* whence our word *algebra,* meaning *restoring,* and the other *mukâbala,* meaning *opposing.* We say *transposing and combining.* Algebra took its name from this phrase. Modern

knowledge starts with Arabic sources. (Tropfke, *Geschichte der Elementar-Mathematik*, Band I, pp. 242–47).

Linear Equations in Two Unknowns

70. Graph of a linear equation in two unknowns. In an equation containing two unknowns, as $2x+4y=1$, either unknown may be regarded as a function of the other. Thus, $y=\frac{1-2x}{4}$, $x=\frac{1-4y}{2}$.

In general, if $ax+by=c$, then $y=\frac{c-ax}{b}$, $x=\frac{c-by}{a}$. Show that y is a linear function of x and that x is a linear function of y.

We have seen, § 8, that a *linear function* is represented graphically by a *straight line*. The straight line representing $y=\frac{c-ax}{b}$ is also said to be the **graph of the equation** $ax+by=c$.

EXERCISES

Graph the following equations:

1. $2x+y=5$

Let $x=0$, then $y=5$; let $y=0$, then $x=2.5$. Plot the points determined by these pairs of numbers and draw the straight line passing through them.

2. $3x+7y=42$
3. $x-18=-3y$
4. $5x=27-2y$
5. $4-3x-y=0$

71. Graphical solution of a system of equations. A *set of values* of the unknowns which satisfies both equations is a **solution** *of the system*. Since the graph of a linear equation in two unknowns is a straight line and since two straight lines either are different and have at most one point in common, or are the same and have all points in common, a system of two *linear* equations cannot have

more than *one solution.* The co-ordinates of the point of intersection of the two straight lines are the *solution* of the system.

EXERCISES

Solve the following systems graphically:

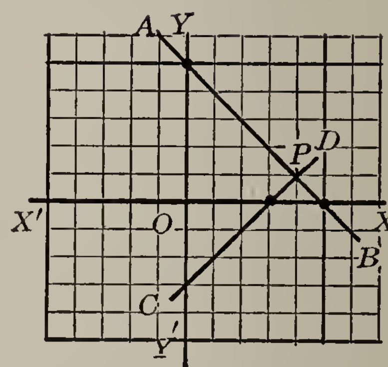


FIG. 51

1. $\begin{cases} x+y=5 \\ x-y=3 \end{cases}$

In $x+y=5$, let $x=0$, then $y=5$; let $y=0$, then $x=5$. These pairs of values of x and y determine two points. The straight line, AB, Fig. 51, passing through these points represents the equation $x+y=5$. Similarly draw CD, the graph of $x-y=3$. The co-ordinates, 4 and 1, of the point of intersection, P, are the required solution, i.e., $(x, y)=(4, 1)$ is the solution of the system $\begin{cases} x+y=5 \\ x-y=3 \end{cases}$.

2. $\begin{cases} x+y=4 \\ y-x=2 \end{cases}$

3. $\begin{cases} 3y+x=14 \\ 2y-5x=-19 \end{cases}$

4. $\begin{cases} 3x+\ y=9 \\ 5x-3y=1 \end{cases}$

5. $\begin{cases} 6x-5y=14 \\ 7x+2y=32 \end{cases}$

72. Solution of a system of linear equations by determinants. In a given system of linear equations in two unknowns the terms containing the unknowns may be brought to one side, and the terms not containing the unknowns to the other side, of the equation. After all similar terms have been combined, the system is of the following *normal form:*

$$\begin{cases} ax+\ by=c \\ a_1x+b_1y=c_1 \end{cases}$$

To eliminate y, the first equation is multiplied by b_1 and the second by b. This gives the equations,

$$\begin{cases} ab_1x+bb_1y=cb_1 \\ a_1bx+b_1by=c_1b \end{cases}$$

GUILLAUME FRANÇOIS ANTOINE L'HÔPITAL

Guillaume François Antoine l'Hôpital

GUILLAUME FRANÇOIS ANTOINE L'HÔPITAL was born at Paris in 1661 and died there in 1704. He was one of the pupils of John Bernoulli, through whom he became one of the earliest appreciators of the infinitesimal calculus, then a science so new that no texts had been written upon it.

He wrote the first treatise on the new science in 1696 under the title *Analyse des infiniment petits.* The wide circulation of this book brought the differential notation of Leibnitz into general use in France and helped to make it known in Europe. He took part in many of the challenges of the friends of Leibnitz and Newton.

L'Hôpital also wrote a treatise on conic sections, which was not published until 1707, three years after the author's death. In this he treated the subject analytically. This means that he used algebraic methods to derive the properties of the conic sections. He did his work so well that his book was regarded as a standard on the subject for nearly a century. He was not a teacher by profession, but through his excellent texts and treatises he became one of the greatest teachers of subsequent times. Mathematical study and textbook writing were his avocation, and his title to fame rests on what he achieved in this avocation.

[See Ball, 5th ed., pp. 369–70.]

Subtracting the second equation from the first and dividing by the coefficient of x,

$$x = \frac{cb_1 - c_1 b}{ab_1 - a_1 b}$$

Similarly,

$$y = \frac{ac_1 - a_1 c}{ab_1 - a_1 b}$$

It will be shown below how these results may be used as formulas to find the solution of the system *directly from the given equations.*

Each of the expressions $cb_1 - c_1 b$, $ab_1 - a_1 b$, and $ac_1 - a_1 c$ is of the form of the difference of two products. Such expressions are called *determinants.**

* Leibnitz in a letter to L'Hôpital, of April 28, 1693, was the first to publish the essential features of the methods of solution of equations by determinants, though his procedure was somewhat different from the modern form. He also drew attention to the importance of the theory of permutations and combinations in determining the factors and signs of the products. Beyond these announcements about the method, Leibnitz did nothing further with it, nor did any of his contemporaries. Aside from a "Note" in a mathematical journal of 1700, nothing further was heard of the method until Gabriel Cramer in an appendix of his book of 1750 on *The Analysis of Curves* solved a system of n equations in n unknowns by the method, showed how to use the theory of combinations and permutations with it, and convinced men of its power.

Bézout (1730–83) and Vandermonde (1735–96) both worked on the theory of determinants and Laplace made important applications of it. Lagrange (1736–1813) applied the doctrine to the problems of analytical geometry and Gauss, in 1801, made important investigations and improvements in the new theory. The modern name *determinants* is due to Cauchy (1789–1857). Jacobi (1804–51) completed the theory of determinants. The classic texts on the subject are Brioschi's of 1854, Baltzer's of 1857, Scott's of 1880, and Muir's of 1882. (Tropfke, I, 143–46.)

The determinant $cb_1 - c_1b$ may be represented by the following symbol:

$$\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}$$

which means that from the product cb_1 we are to subtract the product bc_1.

Similarly $ab_1 - a_1b$ and $ac_1 - a_1c$ may be written

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \text{ and } \begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}$$

Hence the solution of the system

$$\begin{cases} ax + by = c \\ a_1x + b_1y = c_1 \end{cases}$$

takes the form

$$x = \frac{\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}}$$

Notice that the two *denominators* are *the same,* the numbers in the first column being the coefficients of x and the numbers in the second column the coefficients of y *in the given equations.* This makes it easy to remember the *denominator.* The *numerator* of the fraction which gives the value of x is obtained from the *denominator* by replacing the numbers in the first column (the coefficients of x) by the constants c and c_1 respectively. The *numerator* of the fraction which gives the value of y is obtained from the *denominator* by replacing the numbers in the second column (the coefficients of y) by c and c_1.

EXERCISES

Solve the following systems:

1. $\begin{cases} 4x+6y=9 \\ 2x+9y=7 \end{cases}$

$$x=\frac{\begin{vmatrix} 9 & 6 \\ 7 & 9 \end{vmatrix}}{\begin{vmatrix} 4 & 6 \\ 2 & 9 \end{vmatrix}}=\frac{9\cdot 9-6\cdot 7}{4\cdot 9-6\cdot 2}=\frac{81-42}{36-12}=\frac{39}{24}=\frac{13}{8}$$

$$y=\frac{\begin{vmatrix} 4 & 9 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & 6 \\ 2 & 9 \end{vmatrix}}=\frac{4\cdot 7-9\cdot 2}{24}=\frac{10}{24}=\frac{5}{12}$$

Hence, $(x, y)=\left(\frac{13}{8}, \frac{5}{12}\right)$

2. $\begin{cases} 2x+3y=6 \\ 3x-5y=4 \end{cases}$

3. $\begin{cases} 5x+\ \ y=9 \\ 3x+\ \ y=5 \end{cases}$

4. $\begin{cases} 4x+2y=1 \\ 3x-2y=\frac{5}{2} \end{cases}$

5. $\begin{cases} 2x=53\ +y \\ 19x-17y=0 \end{cases}$

6. $\begin{cases} ax-by=c \\ dx-ey=f \end{cases}$

7. $\begin{cases} ax-by=0 \\ x-\ \ y=c \end{cases}$

‡8. $\begin{cases} kx+ly+n=0 \\ 3x-4y=4n \end{cases}$

9. $\begin{cases} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{x}{c}+\frac{y}{d}=1 \end{cases}$

10. $\begin{cases} \frac{3x}{4}-\frac{5y}{6}=1 \\ \frac{5x}{6}-\frac{3y}{4}=2 \end{cases}$

‡11. $\begin{cases} 3x+4my=8mn \\ \frac{x}{7m}-7y+3n=0 \end{cases}$

12. $\begin{cases} ax+\frac{b}{y}=2ab \\ bx+\frac{a}{y}=a^2+b^2 \end{cases}$

13. $\begin{cases} (a+b)x+(a-b)y=4ab \\ (a-b)x-(a+b)y=2a^2-2b^2 \end{cases}$

14. $\begin{cases} a(x+y)+b(x-y)=a \\ (a+b)x-(a-b)y=b \end{cases}$

‡15. $\begin{cases} (a+b)x-(a-b)y=2ac \\ (a+c)x-(a-c)y=2ab \end{cases}$

73. Inconsistent and equivalent equations. We have seen that two linear equations in two unknowns are represented graphically by two straight lines, and that the co-ordinates of the point of intersection form the solution of the system. However, two lines do not always intersect: they may be parallel or they may coincide.

1. If we graph the system $\begin{cases} 2x+y=5 \\ 6x+3y=18 \end{cases}$ we obtain two *parallel* lines, Fig. 52. In this case the equations have *no common solution.* They are said to be **inconsistent, or incompatible.**

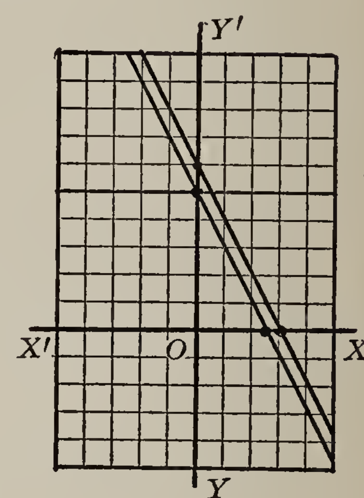


FIG. 52

Moreover, if we solve the same system by determinants, we have

$$x=\frac{\begin{vmatrix} 5 & 1 \\ 18 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix}}=\frac{-3}{0},\quad y=\frac{\begin{vmatrix} 2 & 5 \\ 6 & 18 \end{vmatrix}}{0}=\frac{6}{0}$$

Since it is impossible to divide the numbers -3 and 6 by zero the resulting forms show that the equations have no common solution.

2. If we graph the system $\begin{cases} x-y=2 \\ 5x-5y=10 \end{cases}$ we find that both are represented by the *same* line, Fig. 53. Hence any solution of either equation is a solution of the other. Such equations are said to be **equivalent** or **dependent.**

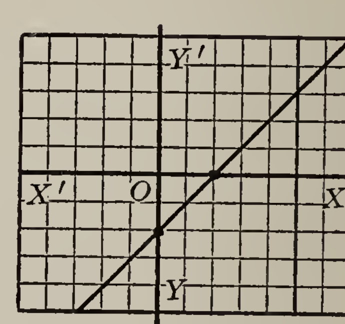


FIG. 53

The difficulty that arises here is not that there is no solution, but that there are too many.

Solving the same system by determinants, we have

$$x=\frac{\begin{vmatrix}2 & -1\\ 10 & -5\end{vmatrix}}{\begin{vmatrix}1 & -1\\ 5 & -5\end{vmatrix}}=\frac{-10+10}{-5+5}=\frac{0}{0},\quad y=\frac{\begin{vmatrix}1 & 2\\ 5 & 10\end{vmatrix}}{0}=\frac{0}{0}$$

Since a number multiplied by zero always gives zero, the expression $\frac{0}{0}$ may represent *any* number. Hence the solution is *indeterminate.* It is easily seen that one equation may be derived from the other by simple multiplication by a constant.

The two preceding examples show that a system of the form

$$\begin{cases} ax+by=c\\ a_1x+b_1y=c_1\end{cases}$$

has one, and only one, solution, if the determinant ab_1-a_1b is *not equal to zero.* Hence this fact may be used to determine whether a system of equations has one and only one common solution.

EXERCISES

Show which of the following systems are equivalent, and which inconsistent.

1. $\begin{cases} 3x+\frac{y}{4}=6\\ 4x+\frac{y}{3}=8\end{cases}$

2. $\begin{cases} 3x-2y=14\\ 9x-6y=36\end{cases}$

3. $\begin{cases} x+\frac{2}{7}y=2\\ \frac{x}{2}+\frac{1}{7}y=1\end{cases}$

4. $\begin{cases} 3x+2y-7-x=12-3y\\ 2x+5y=20\end{cases}$

5. $\begin{cases} 7x-8=4y-2x\\ 18x-8y=16\end{cases}$

6. $\begin{cases} 3x+4y=12\\ 6x+8y=14\end{cases}$

7. $\begin{cases} x-y+1=0\\ 4x+y=16\end{cases}$

Linear Equations with Three or More Unknowns

74. Solution by elimination. Systems of equations in *three* or *more* unknowns are solved by the methods used in solving equations in *two* unknown numbers. In general, the aim should be to obtain first *two* equations in *two* unknowns by *eliminating* the third unknown, and then to solve these two equations.

EXERCISES

Solve the following systems:

1. $\begin{cases} 4x - y + z = 1 \\ x + 2y + 7z = 7 \\ 3x - y - 5z = 5 \end{cases}$

Subtracting the third equation from the first,

$$x + 6z = -4$$

Multiplying the third equation by 2 and adding the resulting equation to the second equation,

$$7x - 3z = 17$$

Solving the system

$$\begin{cases} x + 6z = -4 \\ 7x - 3z = 17 \end{cases}$$

we have $(x, z) = (2, -1)$.

By substituting these values in the first equation, we find $y = 6$.

$\therefore$ $(x, y, z) = (2, 6, -1)$ *is the solution of the system.*

2. $\begin{cases} 2a - b + c = 1 \\ a - 7b - 8c = 1 \\ 7a + 14b + 2c = 7 \end{cases}$

3. $\begin{cases} x + 2y - 4z = 11 \\ 2x = 3y \\ y - 4z = 0 \end{cases}$

4. $\begin{cases} x + 2y - z = 2 \\ 3x - 2y + 2z = 0 \\ 5x - 4y + 3z = 1 \end{cases}$

5. $\begin{cases} 5x - 7y - z = 16 \\ 3x - 2y + 2z = 10 \\ 2x + y + 3z = 6 \end{cases}$

75. Determinant of the third order. The symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a **determinant of the third order.** It represents the following sum:

$$a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-c_1b_2a_3-c_2b_3a_1-c_3a_2b_1.$$

The nine numbers a_1, a_2, a_3, b_1, b_2, b_3, etc., are called the *elements*. The horizontal lines in the square form are the *rows* and the vertical lines the *columns* of the determinant. Each term in the expansion is a product of three elements, no two of which lie in the same row or in the same column.

A determinant of the third order may be expanded as follows:

Draw the diagonal through the first element, a_1, Fig. 54, and the parallels to it through a_2 and a_3 respectively. This gives the terms $a_1b_2c_3$, $a_2b_3c_1$, and $a_3c_2b_1$.

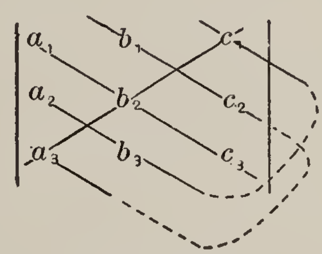


FIG. 54

Then draw the diagonal through c_1 and the parallels through c_2 and c_3. The signs of the last three products are changed. This gives the terms $-c_1b_2a_3$, $-c_2b_3a_1$, and $-c_3a_2b_1$.

EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 5 & 2 & -6 \\ 1 & 4 & 7 \\ 2 & 3 & 1 \end{vmatrix} = 5\cdot 4\cdot 1+1\cdot 3\cdot(-6)+2\cdot 7\cdot 2$

$\qquad -(-6)\cdot 4\cdot 2-7\cdot 3\cdot 5-1\cdot 1\cdot 2$

$= 20-18+28+48-105-2$

$= -29$

2. $\begin{vmatrix} 1 & 3 & 8 \\ -1 & 2 & 0 \\ 1 & -4 & 5 \end{vmatrix}$ **4.** $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 8 \\ 1 & -1 & 0 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 8 & 3 & 0 \end{vmatrix}$ **5.** $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 4 & 3 & 5 \end{vmatrix}$

76. Solution by determinants. By solving the equations

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

a formula may be obtained for the solution of *any* system of three *linear* equations in *three* unknowns.

Eliminating y between the first two equations, we have

$$(a_1b_2-a_2b_1)x+(b_2c_1-b_1c_2)y=d_1b_2-d_2b_1. \tag{1}$$

Eliminating y between the first and third equations, we have

$$(a_3b_1-a_1b_3)x+(c_3b_1-b_3c_1)y=d_3b_1-d_1b_3. \tag{2}$$

Solving equations (1) and (2), we have

$$x=\frac{d_1b_2c_3+d_2b_3c_1+d_3c_2b_1-c_1b_2d_3-c_2b_3d_1-c_3d_2b_1}{a_1b_2c_3+a_2b_3c_1+a_3c_2b_1-c_1b_2a_3-c_2b_3a_1-c_3a_2b_1}$$

According to § 75 this may be written

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Notice that the *denominator* is a determinant whose elements are the coefficients of x, y, and z in the given system and that the *numerator* is derived from the denominator by replacing the coefficients of x by the constants.

Similarly,

$$y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}$$

EXERCISES

Solve by determinants:

1. $\begin{cases} 2x+3y+4z=16 \\ 5x-8y+2z=1 \\ 3x-y-2z=5 \end{cases}$

$$x=\frac{\begin{vmatrix} 16 & 3 & 4 \\ 1 & -8 & 2 \\ 5 & -1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 5 & -8 & 2 \\ 3 & -1 & -2 \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} 2 & 16 & 4 \\ 5 & 1 & 2 \\ 3 & 5 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 5 & -8 & 2 \\ 3 & -1 & -2 \end{vmatrix}}, \quad z=\frac{\begin{vmatrix} 2 & 3 & 16 \\ 5 & -8 & 1 \\ 3 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 5 & -8 & 2 \\ 3 & -1 & -2 \end{vmatrix}}$$

$$\therefore (x, y, z)=(3, 2, 1)$$

2. $\begin{cases} 5x+2y-4z=-3 \\ 4x+5y+2z=20 \\ 3x-3y+5z=12 \end{cases}$

3. $\begin{cases} 3x-y+2z=9 \\ x-2y+3z=2 \\ 2x-3y+z=1 \end{cases}$

4. $\begin{cases} a+3b+9c=23 \\ a+2b+4c=15 \\ a+b+c=9 \end{cases}$

5. $\begin{cases} a+b+c=2 \\ a+3b=4 \\ b-2c=6 \end{cases}$

PROBLEMS AND EXERCISES

77. Solve the following:

1. $\begin{cases} \dfrac{7x+8}{5}-\dfrac{7y-1}{4}=-2 \\ \dfrac{2x-4}{2}+\dfrac{y-1}{3}=-\dfrac{1}{3} \end{cases}$

2. $\begin{cases} 2y-\dfrac{4x-2y}{23-y}=2y-19 \\ 3x+\dfrac{3x-9}{y-18}=3x-17 \end{cases}$

‡3. $\begin{cases} \dfrac{2+x}{3}+\dfrac{2-y}{2}=\dfrac{3(y-4x)}{4} \\ \dfrac{x-3}{2}-5=\dfrac{y+5}{3}-3(y-x) \end{cases}$

4. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=12 \\ \dfrac{2}{x}-\dfrac{3}{y}=14 \end{cases}$

Do not clear of fractions.

Regard $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns.

‡5. $\begin{cases} \dfrac{1}{x-y}+\dfrac{1}{x+y}=15 \\ \dfrac{4}{x-y}-\dfrac{5}{x+y}=17 \end{cases}$

‡6. $\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=4 \\ \dfrac{1}{x}+\dfrac{7}{y}=6 \end{cases}$

7. $\begin{cases} 2x+3y+5=0 \\ 6y+5z=7 \\ 3x+10z=1 \end{cases}$

8. $\begin{cases} x+2y+z=-17 \\ 2x+y-z=-1 \\ 3x-y+2z=2 \end{cases}$

9. $\begin{cases} \dfrac{3}{4x-y}-\dfrac{5}{2x-y}=2 \\ \dfrac{3}{y-2x}+\dfrac{4}{y-4x}=\dfrac{23}{5} \end{cases}$

10. A mixture of alcohol and water contains 10 gallons. A certain amount of water is added and the alcohol is then 30 per cent of the total. Had double the amount of water been added the alcohol would then have been 20 per cent of the whole. How much water was actually added and how much alcohol was there? (Board.)

11. The value of 146 francs is as great as that of 117 shillings. A dollar and 4 francs together are worth 32 cents more than 6 shillings. Find the value in cents of a franc and a shilling. (Board.)

12. A photographer has two bottles of diluted developer. In one bottle 10 per cent of the contents is developer and the

rest water; in the other the mixture is half and half. How much must he draw from each bottle to make 8 oz. of a mixture in which 25 per cent is developer? (Board.)

13. A principal of \$2,500 put at simple interest and for a certain time amounts to \$2,800. If the rate of interest had been 1 per cent higher and the time two years longer, the amount would have been \$3,200. Required the time and rate. (Board.)

14. A certain number of bolts can be bought for a dollar. If 10 more could be bought for a dollar the price would be half a cent less per dozen. What is the price per dozen? (Board.)

15. A man travels 50 mi. in an automobile in $3\frac{1}{4}$ hours. If he runs at the rate of 20 mi. an hour in the country, and at the rate of 8 mi. an hour when within city limits, how many miles of his journey is in the country? (Yale.)

16. A company contracted to make 252 automobiles. Two factories, working together, can make this number in 12 days. Working alone, one factory requires 7 days longer than the other to do this amount. Find the time in which each factory alone can fulfil the contract. (Sheffield.)

17. A and B together can do a piece of work in 12 days. After A has worked alone for 5 days, B finishes the work in 26 days. In what time can each alone do the work? (Sheffield.)

18. Two yachts race over a 48-mile course. Owing to difference in measurement, B is given a start of half a mile in the first trial and is beaten by 6 minutes. In the second trial, the rate of the wind being the same as before, B's start is increased to a mile and a half, and still A wins by 2 minutes. Find the rate in feet per minute of each boat. (Chicago.)

19. A sum of \$1,050 is divided into two parts and invested; the simple interest on the one part at 4 per cent for 6 yr. is the same as the simple interest on the other at 5 per cent for 12 yr.; find how the money is divided. (Princeton.)

20. A man has two sons, one six years older than the other. After two years the father's age will be twice the combined ages of his sons, and six years ago his age was four times their combined ages. How old is each? (Princeton.)

21. In buying coal A gets 1 ton more for $18 than B does; he pays $9 less for 6 tons than B pays. Find the price per ton that each pays. (Princeton.)

22. In paying two bills aggregating $175, a merchant availed himself of discount for cash, 10 per cent on one and 5 per cent on the other, and then paid them both with $166. What was the amount of each bill? (Chicago.)

23. Two locomotives, A and B, are on tracks which cross each other at right angles. When B is at the point of crossing, A has 675 ft. yet to go before reaching this point. In 5 sec. the two locomotives are at an equal distance from the crossing, and in 40 sec. more they are again at an equal distance from it. What is the rate of each in feet per second? Illustrate by a diagram.

24. A dealer has two kinds of coffee, worth 30 and 40 cents per pound respectively. How many pounds of each must be taken to make a mixture of 70 lb. worth 36 cents per pound? (Yale.)

25. A man bought a certain number of eggs. If he had bought 88 more for the same money they would have cost him less by a cent apiece; if he had bought 56 fewer they would have cost more by a cent apiece. How many eggs did he buy and at what price each? (Yale.)

26. An investment at simple interest for 6 yr. amounts to $4,960. If the rate had been 1 per cent greater the amount would have been $5,000 in 5 years. Find the rate and the sum invested. (Chicago.)

27. The sum of the three digits of a number is 16. The sum of the first and third digits is equal to the second; and if the digits in the units and in the tens places be interchanged the resulting number will be 27 less than the original number. What is the original number?

28. A chauffeur engages to accomplish a journey of 105 mi. in a specified time. After traveling 63 mi. uniformly at a rate which will just enable him to keep his agreement, his car is delayed 24 minutes. He then drives $3\frac{1}{2}$ mi. faster per hour than before and arrives exactly on time. What was his original rate? (Board.)

Summary

78. The chapter has taught the meaning of the following terms. Give the meaning of each:

normal form of a linear equation
graph of an equation
solution of a system of equations
determinant of the second order, of the third order
inconsistent equations
equivalent equations

79. Tell how to solve a system of linear equations in two unknowns (1) graphically, (2) by elimination, (3) by determinants.

80. Tell how to solve a system of linear equations in three unknowns (1) by elimination, (2) by determinants.

CHAPTER IV

QUADRATIC EQUATIONS IN ONE UNKNOWN

Methods of Solving Quadratic Equations

81. Quadratic equation. Equations like

$$x^2-5x=6, \qquad \tfrac{1}{2}+5x^2=x, \qquad mx^2-nx=7-3x,$$
$$x^2+ax-abc+a(b+cx+2x^2)=0,$$

are *quadratic equations.*

By bringing all terms to one side of the equation and then collecting similar terms, show that these equations may be changed respectively to:

$$x^2-5x+6=0; \quad 5x^2-x+\tfrac{1}{2}=0; \quad mx^2-(n-3)x-7=0;$$
$$(1+2a)x^2+(a+ac)x+(ab-abc)=0.$$

In general, *any* quadratic equation in one unknown can be changed to the following **normal form:**

$$\boldsymbol{ax^2+bx+c=0,}$$

where a denotes the coefficient of the term in x^2, b the coefficient of the term in x, and c the sum of the terms not containing x, i.e., the sum of the *constant* terms.

Give the values of a, b, and c in each of the equations above.

82. Methods of solving quadratic equations. In §§ 13 and 23 quadratic equations were solved by *graph* and by *factoring.*

The method by factoring has the advantage of being brief and is used when the quadratic function forming the first member is readily factored.

We cannot solve *every* quadratic equation by the graphical method. For example, let us try to use the graphical method to solve the equation:

$$x^2-4x+5=0.$$

Let $$f(x) = x^2 - 4x + 5.$$

The table, Fig. 55, gives the values of $f(x)$ corresponding to integral values of x between -1 and $+5$. It is seen that the graph of $f(x)$ and the $x-$axis have no points in common. Hence the graph does not enable us to find the roots of the equation $x^2 - 4x + 5 = 0$.

x	$f(x)$
-1	10
0	5
1	2
2	1
3	2
4	5
5	10

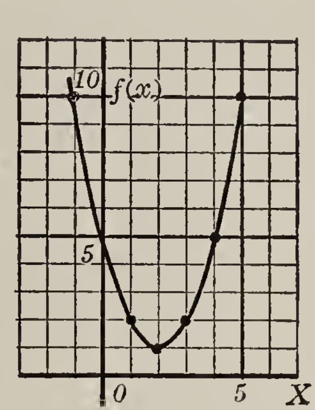


FIG. 55

However, *any* quadratic equation can be solved either by *completing the square* or by the *formula.* These methods are therefore *general.*

83. Solution by completing the square* and by formula. Since every quadratic equation may be changed to the normal form,

$$\boldsymbol{ax^2 + bx + c = 0},$$

we may obtain a solution of *every* quadratic equation by solving

$$ax^2 + bx + c = 0.$$

This will lead to a formula for finding the roots.

Subtracting c from both sides of the equation $ax^2 + bx + c = 0$, and dividing by a, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Completing the square by adding $\frac{b^2}{4a^2}$ to both sides,

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

* This method originated with the Hindus. Aryabhatta (b. 476 A.D.) first used it in a slightly different form from that given here. Brahmagupta (b. 598 A.D.) used it so extensively that it has been given the name Brahmagupta's Rule, and Cridhara later modified it slightly, bringing it to the modern form (Tropfke, Band I, S. 257).

Extracting the square root,

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{4a^2}$$

Hence $x_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $x_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

EXERCISES

Solve the following equations by formula:

1. $3x^2+5x-2=0$

Comparing this equation with the equation $ax^2+bx+c=0$, we find that

$$a=3;\quad b=5;\quad c=-2$$

Substituting these values in the formulas,

$$x=\frac{-5\pm\sqrt{25+24}}{6}=\frac{-5\pm 7}{6}=\frac{1}{3}, \text{ and } -2$$

2. $2x^2+5x+2=0$

3. $2r^2-r-6=0$

4. $1.4x^2+5x=2.4$

5. $6p^2-13p=10p-21$

‡6. $my^2+ny+p=0$

7. $ax^2+(b-a)x-b=0$

8. $a-y^2=(1-a)y$

9. $y^2-1.6y+0.3=0$

Give values correct to two significant figures.

10. $r^2-3.50r+2.80=0$

Give values correct to three significant figures.

‡11. $t^2=.100-.200t$

12. $16.08t^2+20t=1{,}000$

13. $(m-n)y^2-m^2y+m^2n=0$

14. $(2y-1)(y-3)=2$

15. $(x-1(^2(x+3)=x(x+5)(x-2)$

‡16. $y^2-6mry+m^2(9r^2-4n^2)=0$

17. $m^2y^2-(m^2+mn)y=2m^2-5mn+2n^2$

The formula gives:

$$y=\frac{m^2+mn\pm\sqrt{m^4+2m^3n+m^2n^2+8m^4-20m^3n+8m^2n^2}}{2m^2}$$

$$=\frac{m^2+mn\pm\sqrt{9m^4-18m^3n+9m^2n^2}}{2m^2}$$

$$=\frac{m^2+mn\pm(3m^2-3mn)}{2m^2}, \text{ etc.}$$

In solving exercise 17 it is necessary to find the square root of the polynomial $9m^4-18m^3n+9m^2n^2$. This is readily done by inspection.

It is, however, not always easy to find *by inspection* the square root of a polynomial. Hence we shall learn the *process of extracting* the square root of a polynomial before proceeding with the general solution of quadratic equations.

Square Root of Polynomials

84. The process of extracting the square roots of polynomials is suggested by such equations as:

$$\begin{aligned}(a+b)^2&=a^2+2ab+b^2\\(a+b+c)^2&=(a+b)^2+2(a+b)c+c^2\\&=a^2+2ab+b^2+2ac+2bc+c^2\end{aligned}$$

1. We note that a^2, the first term of the polynomial, is the square of the first term of the square root. Therefore, if the polynomial is arranged according to powers of a letter, the *first* term of the root is found by extracting the square root of the first term of the polynomial.

2. The term $2ab$, which is the first of the remaining terms of the polynomial, is twice the product of the first term of the root by the next term. Therefore, the *second* term of the root is found by dividing the first term of the remainder by *twice the first term* of the root.

3. By adding b, the second term of the root, to twice a, the first term, and then multiplying this sum by the second term, b, we obtain $2ab+b^2$. Subtracting this from the first remainder, $2ab+b^2+2ac+2bc+c^2$, we have the second remainder, $2ac+2bc+c^2$.

By dividing $2ac$, the first term of this remainder by $2a$ we find c which is the *third* term of the root. The process in (3) is then continued. If at any time there is no remainder the polynomial is a square.

EXERCISES

Find the square roots of the following polynomials:

1. $4-19n^2+12n-42n^3+49n^4$

First, the polynomial is arranged according to powers of n,

	$7n^2-3n-2=$ Square root
thus: Polynomial $=$	$49n^4-42n^3-19n^2+12n+4$
The *first* term of the root is $\sqrt{49n^4}=7n^2$	
Subtracting the square of $7n^2$ from the polynomial, $(7n^2)^2=$	$49n^4$
we have the remainder:	$-42n^3-19n^2+12n+4$
The first term of the remainder divided by $2\cdot 7n^2$ gives $-3n$, which is the *second* term of the root.	
Adding this to $2\cdot 7n^2$, we have $14n^2-3n$.	
This is multiplied by $-3n$: $(14n^2-3n)(-3n)=$	$-42n^3+9n^2$
The product obtained is then subtracted from the preceding remainder, giving:	$-28n^2+12n+4$
The first term of the last remainder divided by $2\cdot 7n^2$ gives -2, which is the *third* term of the root.	
Adding this to 2 times the sum of the first two terms of the root we have $14n^2-6n-2$.	
This is multiplied by -2: $(14n^2-6n-2)(-2)=$	$-28n^2+12n+4$
The product obtained is then subtracted from the preceding remainder, leaving *no* remainder.	

Hence the given polynomial is a perfect square and the square root has been found exactly.

2. $10x^2+12x^3+1+4x+9x^4$

3. $16x^6+10x-8x^3+1-40x^4+25x^2$

4. $9x^2-3xy^2-30xy+5y^3+25y^2+\frac{y^2}{4}$

‡5. $x^2+9z^2+4y^2-4xy-12yz+6xz$

6. $x^4+25+6x^3-30x-x^2$

‡7. $16x^6+25x^4-24x^5+10x^2-20x^3-4x+1$

8. $12a-23a^4+8a^5+5a^2+4-22a^3+16a^6$

9. $\frac{m^2}{a^2}+11+\frac{6m}{a}+\frac{6a}{m}+\frac{a^2}{m^2}$

‡10. $1+x$, to 5 terms

Solve the following equations:

11. $y^2-4my+ny+3m^2-5mn-2n^2=0$, for y

12. $y^2-3ay-3by+3a=y-9ab$, for y

‡13. $t^2-3at-2=t-2a^2-3a$, for t

14. $x^2-3x-5x^2-2mx^2=3+mx+4m$, for x

Fractional Equations

85. Solve the following equations:

1. $\frac{x+5.4}{x-3.2}+\frac{x-6.3}{x+7.5}=0$ (Yale)

‡2. $\frac{x+a}{x+b}+\frac{x+b}{x+a}=\frac{5}{2}$, for x (Harvard)

3. $\frac{4a^2}{x+2}-\frac{b^2}{x-2}=\frac{4a^2-b^2}{x(4-x^2)}$, for x (Sheffield)

4. $\frac{x+3}{x}-\frac{3-x}{2(x+1)}=\frac{7}{3}$ (Princeton)

5. $\frac{3x}{x-2}-\frac{2}{x+3}+\frac{2}{2-x}=0$ (Board)

6. $\frac{3}{2(x^2-1)}-\frac{1}{4(x+1)}=\frac{1}{8}$ (Princeton)

‡7. $\dfrac{1}{x-3}+\dfrac{1}{x-6}+\dfrac{1}{x-9}=0$ (Harvard)

8. $\dfrac{s-3}{2s}+\dfrac{2s+3}{3}-\dfrac{5s-3}{6}=\dfrac{3s-1}{2}-3$

9. $x+1+\dfrac{x^2}{x^2-1}=\dfrac{x^2}{x+1}-\dfrac{5x-4}{1-x^2}$

‡10. $\dfrac{1}{3(t^2-1)}-\dfrac{2}{t-1}+\dfrac{1}{3(t+1)}=\dfrac{2}{3}$ (Chicago)

11. $\dfrac{9a^3}{x^2}+1-\dfrac{6a\sqrt{a}}{x}=\left(\dfrac{3a-\sqrt{a}}{\sqrt{x}}\right)^2$, for x

‡12. $\dfrac{(a+2b)x}{a-2b}+\dfrac{4b^2}{x}=\dfrac{a^2}{a-2b}$, for x

Problems Leading to Quadratic Equations

86. Solve the following problems:

1. A rectangular box has a volume of 1,500 cubic inches. Its depth is 5 in. and it is 5 in. longer than wide. Find its dimensions. (Sheffield.)

2. One side of a rectangle is 20 cm. longer than the other. The diagonal is 7 cm. longer than the longer side. Find the area of the rectangle. (Harvard.)

‡3. It is shown in plane geometry that the length of the side, s, of a regular decagon inscribed in a circle of radius a is determined from the equation

$$\frac{a-s}{s}=\frac{s}{a}$$

Assuming this equation, solve for s in terms of a.

Find the value of s correct to three significant figures when $a=100$. (Harvard.)

4. On the side AB of a square $ABCD$, a point E is marked at a distance of 10 in. from A. The area of the trapezoid $EBCD$ is less by $22\frac{3}{4}$ sq. in. than three-fourths of the area of the square. How long is a side of the square? (Harvard.)

5. A man, after having bought an article, sells it for \$21. He loses as many per cent as he gave in dollars for the article. What did he pay for it? (Yale.)

6. A trunk 30 in. long is just large enough to permit an umbrella 36 in. long to lie diagonally on the bottom. How much must the length of the trunk be increased if it is to accommodate, diagonally, a gun 4 in. longer than the umbrella? (Chicago.)

7. A rectangular piece of tin is 4 in. longer than it is wide. An open box containing 840 cu. in. is made by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box? (Chicago.)

‡8. An open box, to be made from a square piece of cardboard by cutting out a 4-inch square from each corner and turning up the sides, is to contain 256 cubic inches. How large a square must be used? (Chicago.)

9. The rates of two trains differ by 5 mi. an hour. The faster requires one hour less time to run 280 miles. Find the rate of each. (Yale.)

10. If a body falls from rest, the distance, s, that it falls in t seconds is given by the formula $s=16t^2$. A man drops a stone into a well and hears the splash after 3 seconds. If the velocity of sound in air is 1,086 ft. a second, find the depth of the well.

11. Find the sides of a right triangle in which the sides of the right angle are respectively 20 in. and 10 in. shorter than the hypotenuse.

12. A rectangular tract of land, 800 ft. long by 600 ft. broad, is divided into four rectangular blocks by two streets of equal width running through it at right angles. Find the width of the streets, if together they cover an area of 77,500 square feet. (M.I.T.)

‡13. What is the number of sides of a polygon having 170 diagonals?

14. Two men can do a piece of work in 6 hr. 40 minutes. One can do the work alone in 3 hr. less time than the other. In how many hours can he do it alone?

15. The sum of the two digits of a given number is 5. If the order of the digits is changed, the product of the result by the original number is 736. Find the number.

Equations of Quadratic Form

87. Solve the following equations:

1. $y^4-26y^2+25=0$

2. $x^6-8=7x^3$

3. $4y^2+\frac{4}{y^2}=\frac{97}{9}$

‡4. $6t^4+6=13t^2$

5. $(z^2-2z)^2-7(z^2-2z)+12=0$

‡6. $(4x+5)^2+2(4x+5)-15=0$

Trigonometric Equations

88. Conditional equations containing trigonometric functions of unknown angles are called **trigonometric equations.** For example, $2 \sin x=\tan x$, $5 \sin^2 x+\cos^2 x=2$.

Some trigonometric equations are easily solved by factoring.

If the equation contains *several* trigonometric functions of x, it is generally best to change its form so that it contains *only one* function of x. This is accomplished by use of the following fundamental identities:

1. $\sin x \csc x\equiv 1$
2. $\cos x \sec x\equiv 1$
3. $\tan x \cot x\equiv 1$
4. $\tan x\equiv\frac{\sin x}{\cos x}$
5. $\cot x\equiv\frac{\cos x}{\sin x}$
6. $\sin^2 x+\cos^2 x\equiv 1$
7. $\tan^2 x+1\equiv\sec^2 x$
8. $\cot^2 x+1\equiv\csc^2 x$

EXERCISES

Find all the positive values of the angle between 0° and 360° which satisfy the following equations:

1. $\cot x = 2 \cos x$

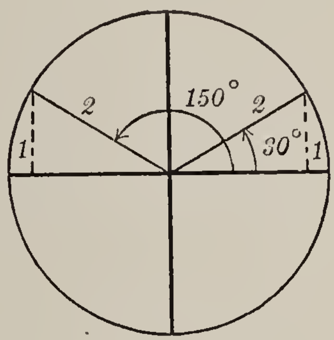


FIG. 56

Since $\cot x = \dfrac{\cos x}{\sin x}$, $\quad \therefore \quad \dfrac{\cos x}{\sin x} = 2 \cos x$

Hence, $\quad \dfrac{\cos x}{\sin x} - 2 \cos x = 0$

Factoring, $\quad \cos x\left(\dfrac{1}{\sin x} - 2\right) = 0$

This equation is satisfied if $\cos x = 0$

or if $\quad \dfrac{1}{\sin x} - 2 = 0$

If $\quad \cos x = 0$, x is equal to 90°, or 270°

If $\quad \dfrac{1}{\sin x} - 2 = 0$, then $\sin x = \frac{1}{2}$

$\therefore \quad x = 30°$, or 150°, Fig. 56.

$\therefore x = 30°$, 90°, 150°, and 270° are the positive values of x, less than 360°, satisfying the given equation.

2. $5 \sin^2 x + \cos^2 x = 2$

Since $\quad \sin^2 x = 1 - \cos^2 x$, it follows that

$5 - 5 \cos^2 x + \cos^2 x = 2$

$\therefore \quad 4 \cos^2 x = 3$

$\cos x = \pm \frac{1}{2}\sqrt{3}$.

If $\quad \cos x = +\frac{1}{2}\sqrt{3}$, it follows that $x = 30°$, or 330°.

If $\quad \cos x = -\frac{1}{2}\sqrt{3}$, it follows that $x = 150°$, or 210°.

$\therefore x = 30°$, 150°, 210°, and 330° are the positive values of x, less than 360°, satisfying the given equation.

3. $\tan x \sec x = \sqrt{2}$

Put $\quad \tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$, $\cos^2 x = 1 - \sin^2 x$

Then show that $\dfrac{\sin x}{1 - \sin^2 x} = \sqrt{2}$

$\therefore \quad \sin x = \sqrt{2} - \sqrt{2} \sin^2 x$

Solve this equation for $\sin x$ and find the required values of x.

4. $2\cos^2 x+3\sin x-3=0$

5. $3\sec^2 x-7\tan^2 x=\tan x$

6. $4\cos^2 x=\cot x$

7. $\tan x=\cos x$

‡8. $\sin^2\theta-\cos\theta+1=0$

9. $2\sin^2\theta+3\cos\theta=0$

‡10. $\cos^2\theta-\sin\theta=\frac{1}{4}$

11. $\tan\theta+\cot\theta=2$

‡12. $\sin^2\theta-2\cos\theta=\frac{1}{4}$

Nature of the Roots of a Quadratic Equation

89. Complex numbers. We have seen, § 82, that some quadratic equations *cannot* be solved by *graph*, e.g., the equation $x^2-4x+5=0$. However, by means of the *quadratic formulas* we find that

$$x=\frac{4\pm\sqrt{16-20}}{2}=\frac{4\pm\sqrt{-4}}{2}=2\pm\sqrt{-1}$$

Thus the roots of the equation $x^2-4x+5=0$ involve the square root of a negative number. We cannot extract the square root, or any *even* root, of a negative number, since the square of a real number is always *positive*. Thus, $\sqrt{-4}$ cannot equal $+2$ or -2, since $(+2)^2=(-2)^2=+4$.

An even root of a negative number is called an *imaginary* number.

Expressions of the form of $a+\sqrt{-b}$, where a is a real number and b a positive real number, are **complex numbers.** Thus, $1+\sqrt{-2}$, $\sqrt{+3}-\sqrt{-8}$ are complex numbers. They are also called **imaginary numbers.** However, the latter term is misleading, because complex numbers are *not imaginary* for the person who has made a study of these numbers.

90. Classification of numbers. *Positive integers* and *fractions* are the first numbers with which the pupil becomes acquainted. Later the study of *negative numbers* is taken up. Positive and negative integers, the

GASPARD MONGE

GASPARD MONGE

GASPARD MONGE, the son of a small peddler, was born at Beaune in 1746 and died at Paris in 1818. A plan of his native town, drawn by him, fell into an army officer's hands, and its excellence so impressed the officer that he recommended that Monge be admitted to the training school at Mézières. His low birth prevented him from receiving a commission in the army, but he was allowed to attend the *annexe* of the school, where he learned surveying and drawing. But he was not sufficiently well born to be allowed to do calculatory problems. A difficult plan for a fortress was to be drawn, and Monge did it by a geometrical construction. This turned the tide of young Monge's fortunes. The officer at first objected to receiving Monge's plan, because he had taken less time than etiquette required for such a problem, but the superiority of Monge's method finally won its acceptance.

In 1768 Monge was made professor of descriptive geometry, though the results of his methods were to be a secret confined to officers above a certain rank.

In 1780 he was made professor of mathematics at Paris, and he communicated his earliest paper of importance to the French Academy in 1781. The paper discussed lines of curvature drawn on a surface. He found that the validity of solutions is not impaired when imaginaries are involved among subsidiary quantities. Euler had treated these questions, but Monge's methods were superior to those of Euler. Monge applied his results to central quadrics in 1795. In 1786 he had published a work on statics.

Monge became embroiled in the politics of the Revolution and narrowly escaped the guillotine. In 1798 he was sent to Italy on state business, and thereafter joined Napoleon in Egypt. After Napoleon's defeat, he escaped to France and settled down at Paris.

Monge was now made professor and gave lectures at the Polytechnic School of Paris on descriptive geometry, and in 1800 published his text entitled *Géomētrie descriptive.* In this he treats the theory of perspective and the theory of surfaces in a masterly way.

On the restoration he was deprived of his offices, stripped of his honors, and thrown out of the French Academy. These humiliations soon led to his death.

[See Ball, pp. 426–27, and Cajori, pp. 286–87.]

fractions and zero, form the domain of **rational numbers.** For example, 4, -8, $\frac{3}{2}$, -1.75 are rational numbers. A rational number may always be expressed *exactly* as the quotient of two integral numbers. This includes integers since they are fractions with 1 as denominator.

However, such numbers as $\sqrt{2}$, $\sqrt{\frac{1}{3}}$, $3\sqrt{7}$ *cannot* be expressed *exactly* as quotients of integers. They are classed as **irrational numbers.** The rational and irrational numbers form the domain of **real numbers.** For example, the number π, known to us from the study of the circle, is a real number, but not a rational number, as it cannot be expressed *exactly* as a quotient of two integral numbers.

91. Graphical representation of real numbers. Positive integers may be represented by equidistant points on a straight line, as OA, Fig. 57, or by the distances of these points from a fixed point, as O.

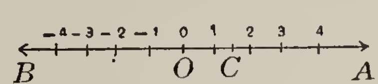


FIG. 57

Negative numbers are then represented by equidistant points laid off in the direction opposite to that of OA, as OB.

The origin, O, represents zero.

Fractions are represented by intermediate points. Thus the point C represents the fraction $\frac{3}{2}$. Similarly *any rational* number may be represented by a point on the line.

Although between any two rational numbers, however close, other rational numbers may be inserted, there are points on the line which do *not* represent *rational* numbers. For example, we know that $\sqrt{2}$ is the length of the diagonal of a square whose side is of unit length. Therefore, by laying off on OA a length equal to this diagonal, we obtain a single definite point which represents

the irrational number $\sqrt{2}$. Indeed it can be shown that to *any irrational* number there corresponds a definite point on OA.

Hence any real number can be represented by a point on a straight line. This line is called the *axis of real numbers*.

92. Nature of the roots of a quadratic. The character of the roots of the equation $ax^2+bx+c=0$, depends upon the number b^2-4ac. The function b^2-4ac is called the **discriminant** of the equation $ax^2+bx+c=0$. In the following we consider the coefficients a, b, and c to be rational numbers.

1. If $b^2-4ac=0$, the two values of x are the same. Thus the roots of the quadratic are *real, rational,* and *equal.*

For example, for the equation $x^2-6x+9=0$, we have $b^2-4ac=36-36=0$.

Hence, without solving the equation, we know that the roots are real, rational, and equal.

2. If $b^2-4ac>0$, the roots are *real* and *unequal.*

(1) If b^2-4ac is a square, the roots are *rational.*

(2) If b^2-4ac is not a square, they are *irrational.*

For example, for the equation $x^2-9x+14=0$, we have $b^2-4ac=81-56=25$.

Hence the roots are real, rational, and unequal.

For the equation $x^2+5x+1=0$ we have

$$b^2-4ac=25-4=21.$$

Hence the roots are irrational.

3. If $b^2-4ac<0$, the expression $\sqrt{b^2-4ac}$ is imaginary and the roots of the equation are called *complex.*

Thus for $2x^2+x+1=0$ we have $b^2-4ac=1-8=-7$.

Hence the roots of this equation are complex.

The preceding discussion may be represented in a table as follows:

$$\text{The roots of a quadratic equation are}\begin{cases}\text{complex, if } b^2-4ac<0\\[2ex] \text{real, if } b^2-4ac \text{ is not } <0 \begin{cases}\text{rational and}\begin{cases}\text{equal, if } b^2-4ac=0\\ \text{unequal, if } b^2-4ac>0 \text{ and a perfect square}\end{cases}\\[2ex] \text{irrational, if } b^2-4ac>0\text{, and } \textit{not} \text{ a perfect square}\end{cases}\end{cases}$$

EXERCISES

Without solving, determine the nature of the roots of each of the following equations:

1. $3x^2-8x+5=0$

$b^2-4ac=64-4(3)(5)=64-60=4$

$\therefore$ the roots are real, rational, and unequal.

2. $x^2-4x+8=0$

3. $a^2+3a-1=0$

4. $5x^2-3x=2$

5. $3x^2=7x+6$

6. $9x^2+12x+4=0$

7. $7y^2+3y=0$

8. $5x^2+7x+3=0$

9. $x^2-6x+4=0$

Find the values of d for which the roots of the following equations are equal:

10. $9x^2+(1+d)x+4=0$

$b^2-4ac=1+2d+d^2-144$

Hence, to make the roots equal, we put

$d^2+2d-143=0$

$\therefore d=12, \ -14$

11. $y^2+y+d=0$

12. $2x^2+(1+d)x+2=0$

13. $x^2-4dx+4=0$

‡**14.** $y^2+3dy+d+7=0$

15. $(d+1)x^2+dx+d+1=0$

‡**16.** $2dx^2+(5d+2)x+(4d+1)=0$

Relation between the Roots and the Coefficients of a Quadratic

93. Denoting the roots of the equation $ax^2+bx+c=0$ by

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } r_2=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

we have by addition:

$$r_1+r_2=\frac{-2b}{2a}=-\frac{b}{a};$$

by multiplication:

$$r_1 \cdot r_2=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{2a \cdot 2a}=\frac{b^2-b^2+4ac}{4a^2}=\frac{c}{a}.$$

Hence *the sum of the roots is* $-\frac{b}{a}$

and the product of the roots is $\frac{c}{a}$

EXERCISES

Find the sum and the product of the roots of the equations in exercises 1 to 5.

1. $2x^2-9x+8=0$

Since $a=2$, $b=-9$, $c=8$, it follows that

$$r_1+r_2=-\frac{-9}{2}=\frac{9}{2}$$

and that

$$r_1r_2=\frac{8}{2}=4.$$

2. $2x^2-9x-5=0$

3. $x^2-12x-13=0$

4. $x^2+2x+2=0$

5. $4-y-6y^2=0$

6. One root of the equation $x^2-kx+21=0$ is 7. Find the value of k.

We have
$$\begin{cases} r_1+r_2=k \\ r_1\cdot r_2=21 \\ r_1 \qquad =7 \end{cases}$$
$$\therefore\ r_2=3 \text{ and } k=10$$

7. One root of the equation $3x^2-kx+10=0$ is 5. Find the other.

8. Find the values of p and q in the equation $x^2+px+q=0$,

1. If the roots are 6 and -4,
2. If the roots are $3-\sqrt{6}$ and $3+\sqrt{6}$.

9. Form the equation whose roots are 2 and -3.

Since $2+(-3)=-\frac{b}{a}$ and $2(-3)=\frac{c}{a}$, we may substitute these values in the equation $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

This gives $$x^2+x-6=0.$$

10. Form the quadratic equations whose roots are:

1. $-3,\ 10$
2. $-8,\ -3$
3. $6,\ -\frac{1}{5}$
4. $\frac{1}{2},\ \frac{3}{4}$
5. $2\pm\sqrt{3}$
6. $a,\ -b$
7. $\sqrt{3},\ -\sqrt{3}$
8. $-m+n,\ -m-n$

Factoring

94. The solution of a quadratic equation enables us to find the *factors* of the quadratic trinomial ax^2+bx+c.

Show that
$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$=a[x^2-(r_1+r_2)x+r_1r_2]$$
$$\therefore\ \boldsymbol{ax^2+bx+c=a(x-r_1)(x-r_2)}.$$

EXERCISES

Determine the factors of the quadratic functions given in exercises 1 to 7.

1. $5x^2+3x-20$

Let $5x^2+3x-20=0$

$$\text{Then } r_1=\frac{-3+\sqrt{409}}{10}, \quad r_2=\frac{-3-\sqrt{409}}{10}$$

$$\therefore \quad 5x^2+3x-20=5\left[x-\frac{-3+\sqrt{409}}{10}\right]\left[x-\frac{-3-\sqrt{409}}{10}\right]$$

$$=\tfrac{1}{20}(10x+3-\sqrt{409})(10x+3+\sqrt{409})$$

2. $2x^2+5x-7$

3. $2x^2+5x+2$

4. $6y^2+y-1$

5. $x^2-2ax+a^2-b^2$

‡**6.** $x^2+4abx-(a^2-b^2)^2$

‡**7.** $abx^2+(a+b)x+1$

Reduce the following fractions:

8. $\dfrac{2y^2+7y-4}{3y^2+11y-4}$

9. $\dfrac{2t^2+8t-90}{3t^2-36t+105}$

‡**10.** $\dfrac{ax^2-2ax+x+a-1}{ax^2-3ax+x+2a-2}$

‡**11.** $\dfrac{20x^2y^2+20xyz-21z^2}{10x^2y^2+24xyz-18z^2}$

Summary

95. The chapter has taught the meaning of the following terms:

quadratic equation
normal form of a quadratic equation
imaginary numbers
complex numbers
real numbers
rational numbers
irrational numbers
nature of the roots of a quadratic, discriminant

96. The following are typical problems of the chapter:

1. To solve quadratic equations in one unknown by the following methods:

   (1) By factoring,
   (2) By formula,
   (3) By completing the square,
   (4) By graph.

2. To extract the square root of a polynomial.

3. To solve fractional equations which reduce to quadratic form.

4. To solve equations of other degree than the second, but of the form of quadratics.

5. To solve trigonometric equations.

6. To solve problems leading to quadratic equations.

7. To represent real numbers graphically.

8. To determine the nature of the roots of a quadratic equation.

9. To solve quadratic equations, having given a relation between the roots and the coefficients.

10. To factor quadratic trinomials by means of the quadratic formula.

CHAPTER V

FACTORING. FRACTIONS

97. Purpose of the chapter. The work in factoring given in §§ 22 and 94 completes what is commonly taught about this subject in an elementary course in mathematics. We have had *no general* method by which *all* polynomials can be factored, but we have learned to recognize certain forms which suggest a particular method to be used in factoring them. It is the purpose of this chapter to review and summarize what we should know about factoring and to make use of factoring in the operations with algebraic fractions.

98. The difference of two squares.

Find the prime factors of the following:

1. x^2-4y^2

$x^2-4y^2=(x+2y)(x-2y)$

2. $9a^2-25$

3. $1-r^4$

4. $9a^2x^6-4b^2y^2$

5. a^6-b^6, a^8-b^8

6. a^4-b^4, $a^{12}-b^{12}$

7. $16a^2-(2m-3n)^2$

8. $(x^2-y^2)^2-x^6$

9. $9c^2-(3a-2c)^2$

10. $36(a+b)^2-25(c-d)^2$

11. $(a+b)^2-(a-b)^2$

12. $(4a+5)^2-(2x-3)^2$

13. $(x+y+z)^2-(x-y-z)^2$

Reduce the following fractions to simplest terms:

14. $\dfrac{m(x^2-y^2)}{(a+b)(x-y)}$

15. $\dfrac{(x+a)(x-b)}{(x^2-a^2)(x^2-b^2)}$

16. $\dfrac{x^2+y^2}{x^4-y^4}$

17. $\dfrac{(4a-d)^2-(2b-3c)^2}{(4a-2b)^2-(3c-d)^2}$

Add and subtract as indicated:

18. $\frac{x}{2x-3}-\frac{x}{2x+3}+\frac{2x-6}{9-4x^2}$

19. $3x+\frac{2}{x-3}-\frac{3x^2}{x+3}+\frac{4}{x^2-9}$

Multiply and divide as indicated:

20. $\frac{(a+8)^2}{(a-4)^2}\cdot\frac{a^2-16}{a^2-64}$

21. $\frac{x^2-9}{x-4}\div\frac{(x+3)^2}{(x-4)^2}$

99. The sum or differences of like powers.

Factor the following:

1. $64a^3+27b^3$

$64a^3+27b^3=(4a+3b)(16a^2-12ab+9b^2)$

2. $8x^3+125y^3$
3. $27x^3-8y^3$
4. $343+x^3$
5. $p^3-\frac{1}{8}$
6. $512c^3+27d^3$
7. $8v^{18}+27w^{18}$
8. $(a+b)^3-c^3$
9. $(w+3)^3+a^3$
10. $x^3-(m+n)^3$
11. $(5m-n)^3+c^3$

Reduce to the simplest form:

12. $\frac{x^3-1}{x^2+x+1}$

13. $\frac{64a^6b^6+1}{4a^2b^2+1}$

Multiply as indicated:

14. $\frac{x^4-y^2}{x^3-y^3}\cdot\frac{x^2-y^2}{x^3+y^3}\cdot\frac{x^6-y^3}{x^2+y^2}$

15. $\frac{x^2-9}{x^3-27}\cdot\frac{x^2+3x+9}{x+3}$

Subtract as indicated:

16. $\frac{a^2+a}{a^3-1}-\frac{1}{a-1}$

17. $\frac{3}{a}-\frac{5}{a-1}-\frac{2a-3}{a^2-1}$

Solve for x:

18. $\frac{2x-3}{x^2-1}+\frac{x}{x-1}=\frac{x}{x+1}$

Factor the following:

19. a^6+b^6; a^9-b^9; a^9+b^9; $a^{12}+b^{12}$

100. Trinomials.

Factor the following:

1. $4x^2-12xy+9y^2$
2. $x^4y^2+2x^2yz+z^2$
3. $5x^2-38x+21$
4. $6b^2-29b+35$
5. $1-6xy+5x^2y^2$
6. $2x^2+11x+12$
7. $x^4-3x^2y^2+y^4$
8. $a^4+2a^2b^2+9b^4$
9. $6x^2-17x+5$
10. $3x^2+8x-7$
11. $(u+v)^2+4t(u+v)+4t^2$
12. $9k^2+6k(r+s)+(r+s)^2$

Reduce the following fractions to lowest terms:

13. $\dfrac{x^2-3x+2}{x^2-4x+3}$
14. $\dfrac{2x^2+5x+3}{3x^2+7x+4}$
15. $\dfrac{2a^2+17a+21}{3a^2+26a+35}$
16. $\dfrac{2m^2+5m+2}{2m^2+7m+3}$

Perform the indicated operations:

17. $\dfrac{b^2+b-6}{2b^2+5b-3}-\dfrac{b^2+8b+15}{2b^2+7b-15}$
18. $\dfrac{10bx+3b^2+3x^2}{10bx-3b^2-3x^2}\div\dfrac{(3b+x)x}{(x-3b)b}$
19. $\dfrac{4a^3-9a}{a-7}\cdot\dfrac{a^2-2a-35}{2a^3-3a^2}$

Solve the following equations:

20. $\dfrac{2}{3+2x}-\dfrac{3-4x}{9-4x^2}-\dfrac{3}{9-12x+4x^2}=0$
21. $\dfrac{2}{x+4}+\dfrac{2-x}{x^2-16}-\dfrac{2+x}{x^2-8x+16}=0$

101. Polynomials.

Factor the following polynomials:

1. $16a^2b^2+48a^2b-16ab+8a$
2. $6ax^2y-8ax^3-5ay^2-4axy$
3. x^4+x^3-x-1
4. $a^2+b^2-c^2-2ab$

5. a^2-b^2+a+b
6. $x^4+4x^3-8x-32$
7. $m^2+6m-x^2+9-4xy-4y^2$
8. $12y^3+3y^2-8y-2$
9. a^3+b^3+a+b
10. $x^2y+y^2z+xz^2-x^2z-xy^2-yz^2$
11. $8x^3+36x^2y+54xy^2+27y^3$
12. $3x^3-2x^2+5x+6$
13. $x^3+9x^2+10x+2$
14. $x^3-6x^2+11x-6$

Reduce to lowest terms:

15. $\dfrac{x^2+ax+bx+ab}{(x^2-a^2)(x^2-b^2)}$
16. $\dfrac{x^2+ax+bx+ab}{x^2+3ax+2a^2}$
17. $\dfrac{2ax-6x-2ay+6y}{3akx-9kx-3aky+9ky}$
18. $\dfrac{2ax+3bx+4a+6b}{x^2+x(b+2)+2b}$

Perform the indicated operations:

19. $\dfrac{x^2+y^2+2xy-z^2}{z^2-x^2-y^2+2xy}\div\dfrac{x+y+z}{y+z-x}$
20. $\dfrac{3a-3b}{5c-5d}\cdot\dfrac{cx-dx+cy-dy}{am-bm+an-bn}$
21. $\dfrac{1}{ab+ac-bk-ck}+\dfrac{1}{bm+cm+bk+ck}$

Miscellaneous Exercises on Factoring

102. Factor the following:

1. x^5-1
2. $16x^2+25y^2+40xy$
3. $6x^2+11x-10$
4. $x^4+x^2y^2+y^4$
5. $x^4+x^3y-xy^3-y^4$
6. x^3-7x+6
7. $a^3c^3+b^3$
8. $32a^2-29ab+5b^2$
9. $\dfrac{x^2}{y^2}-3\dfrac{y^2}{x^2}+2$
10. $(x^2-5x)^2-2(x^2-5x)-24$
11. $x(x+1)(4x-5)-6(x+1)$
12. $4x^4+y^4-5x^2y^2$
13. $a^4-14a^2b^2+b^4$
14. $64x^{6a}-y^{12a}$
15. $a^2+b^2+2ab+8a+8b-9$
16. a^4+4
17. $a^4-a^2b^2+b^2-1$
18. $(a+b)^2-(a^2-b^2)-6(a-b)^2$

Miscellaneous Exercises on Fractions

‡**103.** Solve the following:

1. Simplify:

$$\left\{\left(\frac{a^3}{c^3}-\frac{c^3}{a^3}\right)\div\left(\frac{a}{c}+\frac{c}{a}\right)\right\}\times\left\{c^2-\frac{c^4}{c^2-a^2}\right\}\cdot \quad \text{(Sheffield)}$$

2. Simplify:

$$\left\{\frac{a^6+c^6}{a^3c^3}-\left[\frac{a}{c}+\frac{c}{a}\right]\right\}\div\left\{\frac{a^2}{c^2}-\frac{c^2}{a^2}\right\}$$

Check by substituting $a=2$, $c=1$ in the original fractions and in the result. (Sheffield)

3. Simplify:

$$\frac{m^2+mn}{m^2+n^2}\times\frac{m^3-mn^2-m^2+n^2}{m^3n-n^4}\times\frac{m^2n^2+mn^3+n^4}{m^4-2m^3+m^2}\div\frac{m^3n+2m^2n^2+mn^3}{m^4-n^4} \quad \text{(Chicago)}$$

4. Subtract as indicated:

$$\frac{r+s}{(r-t)(s-t)}-\frac{s+t}{(r-s)(t-r)}-\frac{r+t}{(t-s)(s-r)} \quad \text{(Chicago)}$$

5. Reduce to the simplest form:

$$\frac{a^2-4ax-21x^2}{a^2-49x^2} \qquad \frac{ax+2by+2bx+ay}{x^3+3x^2y+3xy^2+y^3}$$

$$\frac{5am+10an}{2m^2+5mn+2n^2} \qquad \frac{1}{xy+y^2}-\frac{1}{x^2+xy}$$

$$\frac{(a^2-c^2)x^2+2ax+1}{ax-cx-1} \qquad \frac{a^3-2a^2-a+2}{a^2-1}$$

$$\frac{x}{(x-y)(x-z)}+\frac{y}{(y-x)(y-z)}+\frac{z}{(z-x)(z-y)} \quad \text{(Chicago)}$$

6. Simplify:

$$\frac{\frac{3}{2}}{x}-\left[\frac{x-1}{x}+\frac{1}{2}\left(\frac{x-1}{x+1}-\frac{(x-2)(x-3)}{x(x+1)}\right)\right] \quad \text{(Princeton)}$$

7. Simplify:

$$\left(\frac{x}{1+x}+\frac{1-x}{x}\right)\div\left(\frac{x}{1+x}-\frac{1-x}{x}\right) \quad \text{(Harvard)}$$

8. Simplify:

$$\left(a^2+\frac{b^4}{a^2-b^2}\right)(a^2+b^2)\div\left(\frac{a}{a+b}+\frac{b}{a-b}\right) \quad \text{(Sheffield)}$$

9. Reduce:

$$\frac{1-2x}{1-x^3}\div\frac{1-2x+x^2-2x^3}{1+2x+2x^2+x^3} \quad \text{(Sheffield)}$$

10. Reduce:

$$\frac{1+n-n^3-n^4}{1-a^2}\div\frac{n^2-1}{a^2-1} \quad \text{(Sheffield)}$$

11. Simplify:

$$\left(5-\frac{a^2-19x^2}{a^2-4x^2}\right)\div\left(3-\frac{a-5x}{a-2x}\right) \quad \text{(Yale)}$$

12. Simplify:

$$1-\left\{\frac{c^3+y^3}{(c-y)^2}\div\left[\frac{c^4+c^2y^2+y^4}{c^3-y^3}\times\frac{(c+y)^2}{c^2-y^2}\right]\right\} \quad \text{(Board)}$$

13. Simplify:

$$\frac{8c^3-1}{9c^2-12c+4}\cdot\left(1-\frac{4}{3c+2}\right)\div\left(\frac{2c-1}{9c^2-4}\right) \quad \text{(Board)}$$

14. Simplify:

$$\left\{\frac{x+y}{x-y}-\frac{x-y}{x+y}+\frac{4y^2}{y^2-x^2}\right\}\div\left(\frac{x-y}{x+y}-1\right) \quad \text{(Board)}$$

15. Find the highest common factor and the least common multiple of x^3-3x^2+x-3
and x^3-3x^2-x+3 (Columbia)

Miscellaneous Exercises on Complex Fractions

‡104. Solve the following:

1. Simplify the following expression:

$$\frac{\dfrac{a}{x-a}-\dfrac{x-a}{a}}{\dfrac{1}{x+a}+\dfrac{2a}{x^2-a^2}} \quad \text{(Harvard)}$$

2. Reduce the following expression to a single fraction:

$$\frac{a}{b^2}-\frac{a}{b^2+\dfrac{cb}{a-\dfrac{c}{b}}} \quad \text{(Harvard)}$$

3. Solve:

$$\left(\frac{1+x}{1-x}-\frac{1-x}{1+x}\right)\left(\frac{3}{4x}+\frac{x}{4}-x\right)=\frac{\left(x-3+\dfrac{5x}{2x-6}\right)\dfrac{3}{2}x}{2x-1+\dfrac{15}{x-3}} \quad \text{(Yale)}$$

4. Reduce to the simplest form:

$$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2-x^2}}{\dfrac{1}{a+x}-\dfrac{1}{a-x}-\dfrac{2a}{a^2-x^2}} \quad \text{(Chicago)}$$

5. Reduce:

$$\frac{\dfrac{a^2}{x^2}-1}{\dfrac{a^2-2ax}{x^2}+1} \quad \text{(Chicago)}$$

6. Reduce:

$$\frac{\dfrac{1}{1-x}+\dfrac{1}{1+x}}{\dfrac{2}{1-x^4}}; \quad \frac{2+ac+\dfrac{1}{ac}}{a^2c^2-1} \quad \text{(Chicago)}$$

7. Simplify:

$$\frac{x^4+x^3-x-1}{1-y^2}\cdot\frac{y^2-1}{x^2-x}\left(1-\frac{1}{1-\dfrac{1}{x}}\right) \quad \text{(Princeton)}$$

8. Simplify:

$$\frac{6a^3+7ab^2+12b^3}{3a^2-5ab-4b^2}-\frac{1}{\frac{3}{19b}-\frac{5a+4b}{19a^2}} \quad \text{(Harvard)}$$

9. Solve:

$$\frac{x-\frac{5}{3}}{\frac{5}{3}(x-1)}+\frac{\frac{7}{3}-x}{\frac{7}{3}(1+x)}=\frac{1}{35\left(1-\frac{1}{x}\right)}+\frac{1}{7}$$

10. Solve:

$$1+\frac{1}{a+x}=\frac{a}{x+2a+\frac{a^2}{x}} \quad \text{(Harvard)}$$

11. Solve for z:

$$\frac{c+\frac{z}{c-d}}{c-\frac{z}{c+d}}=\frac{2c}{1+d} \quad \text{(Chicago)}$$

12. Show that the equation

$$\frac{2ax}{x+2a^2+6a}=\frac{1}{\frac{x}{a^2+x^2-9}+\frac{a+3}{x}}$$

reduces to

$$x^2(x^2-2ax+a^2-9)=0 \quad \text{(Harvard)}$$

13. Simplify:

$$\frac{\frac{2y}{x+2y}-\frac{x}{2y-x}+\frac{8y^2}{x^2-4y^2}}{\frac{4y-x}{(2y-x)^2}}; \quad 1-\frac{\left(1-\frac{c^4}{9}\right)-\left(1-\frac{c^4}{16}\right)}{1-\frac{7c^4}{144}} \quad \text{(Yale)}$$

14. Simplify:

$$\frac{\frac{a^2+b^2}{b}-a}{\frac{1}{b}-\frac{1}{a}}\cdot\frac{a^2-b^2}{a^3+b^3}\cdot\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\cdot\left(\frac{a}{a+b}+\frac{b}{a-b}\right)$$

15. Simplify:

$$\frac{x^2}{a+\frac{x^2}{a+\frac{x^2}{a}}}$$

16. Simplify:

$$\frac{\left(\frac{x+1}{x-1}\right)^2-2+\left(\frac{x-1}{x+1}\right)^2}{\left(\frac{x+1}{x-1}\right)^2-\left(\frac{x-1}{x+1}\right)^2} \quad \text{(Cornell)}$$

Summary

105. Polynomials to be factored may be classified according to the number of terms they contain.

I. Binomials

1. *The difference of two squares:*
   (1) $x^2-y^2=(x+y)(x-y)$
   (2) $(a+b)^2-(s+t)^2$
   $=(a+b+s+t)(a+b-s-t)$
2. *The difference of two cubes:*
   $x^3-y^3=(x-y)(x^2+xy+y^2)$
3. *The sum of two cubes:*
   $x^3+y^3=(x+y)(x^2-xy+y^2)$
4. *The sum or difference of like powers higher than the fourth:*
   x^5-y^5; x^5+y^5; x^7-y^7; x^7+y^7; etc.

II. Trinomials

1. *The trinomial of the form ax^2+bx+c, factored by trial:*
   (1) $10x^2-17x+3=(2x-3)(5x-1)$
   (2) $ax^2+bx+c=a(x-r_1)(x-r_2)$, r_1 and r_2 being the roots of the equation $ax^2+bx+c=0$
2. *The trinomial square:*
   $x^2\pm 2xy+y^2=(x\pm y)^2$
3. *The incomplete trinomial square:*
   $x^4+x^2y^2+y^4=(x^4+2x^2y^2+y^4)-x^2y^2$
   $=(x^2+y^2+xy)(x^2+y^2-xy)$

III. Polynomials, not including the forms given in I and II

1. *Polynomials containing a common monomial factor:*

   $ax+ay+az=a(x+y+z)$

2. *The perfect cube of a binomial:*

   $a^3 \pm 3a^2b+3ab^2 \pm b^3=(a \pm b)^3$

3. *Polynomials whose terms may be so grouped as to change them to one of the preceding forms:*

   (1) $x^2+2xy+y^2-k^2=(x+y)^2-k^2$

   (2) $x^2+2xy+y^2-a^2+2ab-b^2$
   $=(x+y)^2-(a-b)^2$

   (3) $x^2+2xy+y^2-5x-5y+6$
   $=(x+y)^2-5(x+y)+6$

4. *Polynomials containing binomial factors of the form $x \pm a$:*

   $3x^3-x^2-4x+2=(x-1)(3x^2+2x-2)$

CHAPTER VI

EXPONENTS. RADICALS. IRRATIONAL EQUATIONS

The Fundamental Laws of Positive Integral Exponents

106. Base. Exponent. Power. The symbol a^3 means $a \cdot a \cdot a$. Similarly, a^n means $a \cdot a \cdot a \cdot a \ldots$ (n factors).

The number a in a^n is the **base,** n is the **exponent** and a^n is the nth power of a or, briefly, a^n is a **power.** Accordingly, a^n has a meaning only if n is a *positive integer.*

107. The product of two powers having equal bases. The product of two powers having the same base may be simplified as shown in the following illustrations:

1. $5^3 \cdot 5^4 = (5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5) = 5^7$

2. $(-2)^2 \cdot (-2)^3 = (-2)(-2)(-2)(-2)(-2) = (-2)^5$

3. $a^m \cdot a^n = [a \cdot a \cdot a \ldots (m \text{ factors})]\ [a \cdot a \cdot a \ldots (n \text{ factors})]$
$= [a \cdot a \cdot a \cdot a \ldots (m+n) \text{ factors}]$
$= a^{m+n}$

$$\therefore\ \boldsymbol{a^m \cdot a^n = a^{m+n}}$$

Express this law in words.

EXERCISES

Write the following expressions in simplest form:

1. $a^9 \cdot a$

2. $(-a)^9 \cdot (-a)$

3. $b^{2a} \cdot b^a$

4. $a^{n+1} \cdot a^2$

5. $x^{3r} \cdot x^r$

6. $a^x b^x \cdot a^{2x} b^{3x}$

7. $2a^3 \cdot 3a^2$

8. $(2a)^3 \cdot 3a^2$

9. $(-b)^3 - (-b)^5$

10. $(-2)^3 + (-5)^2 - (-1)^4$

11. $2(x+y)^2 \cdot 3(x+y)^3$

12. $(a-b)^n \cdot (a-b)^{2n+3}$

13. $3a^{2n+3}b^{n-4} \cdot 4a^{n-6}b^{3n-2} \cdot a^3b^{4n}$

14. $a^2(a+b)^5(a-b)^{r-1} \cdot a^r(a+b)^{r+1}$

Find the value for $x=-2$:

15. $3x^3-2x^2+5x-4$

16. $x^4+2x^3-7x^2-3x+2$

Factor the following polynomials:

17. $x^{4m}-5x^{2m}+6$

18. $x^{m+4}+2x^{m+2}+10x^m$

108. The quotient of two powers having equal bases. The following examples illustrate how to find the quotient of two powers with equal bases:

1. $\dfrac{a^5}{a^3}=\dfrac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a\cdot a}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}}=a^2$

2. $\dfrac{(-3)^6}{(-3)^4}=\dfrac{(-\cancel{3})(-\cancel{3})(-\cancel{3})(-\cancel{3})(-3)(-3)}{(-\cancel{3})(-\cancel{3})(-\cancel{3})(-\cancel{3})}=(-3)^2$

3. $\dfrac{a^m}{a^n}=\dfrac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\ldots.(m\text{ factors})}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\ldots.(n\text{ factors})}=a\cdot a\cdot\ \ldots.[(m-n)\text{ factors}]$

$$=a^{m-n}$$

$$\therefore \frac{a^m}{a^n}=a^{m-n};\text{ provided } m>n.$$

Express this law in words.

EXERCISES

Write the following expressions in simplest form:

1. $x^{10}\div x$

2. $\dfrac{m^{12}}{m^2}$

3. $(-a)^6\div(-a)^2$

4. $\dfrac{a^{x+4}}{a^3}$

5. $\dfrac{(a+b)^{3x+y}}{(a+b)^{x+y}}$

6. $\dfrac{a^{2n}}{a^2}$

7. $\dfrac{a^2\cdot a^{2n-1}}{a^n}$

8. $\dfrac{(a^2-b^2)(x^3+y^3)^2}{(x+y)^2(a+b)^2}$

9. $\dfrac{x^2y^2(x-y)^4}{(x^2-y^2)(x-y)^3}$

10. $\dfrac{9a^6b^3c^4}{4x^4y^2z}\div\dfrac{3a^2bc^3}{8x^5y^2z^3}$

11. $\dfrac{x^4y^7z^2+x^3y^5z}{xyz}$

109. The power of a product. The following illustration shows how to find the power of a product:

$$\begin{aligned}(ab)^3 &= ab \cdot ab \cdot ab\\ &= a \cdot a \cdot a \cdot b \cdot b \cdot b\\ &= a^3b^3\end{aligned}$$

Similarly, show that $(2 \cdot 5)^3 = 2^3 5^3$; $(3 \cdot a)^4 = 3^4 a^4$.

Show that $(ab)^m = a^m b^m$, or $a^m b^m = (ab)^m$.

Express this law in words.

EXERCISES

Express the following powers as products:

1. $(2ab)^4$
2. $(xy)^3$
3. $(3xyz)^4$
4. $(-2ab)^2$
5. $(abxy)^{2n}$
6. $(2mn \cdot 3p)^3$

Find the value of each of the following:

7. $2^3 \cdot 3^3$

$$\begin{aligned}2^3 \cdot 3^3 &= (2 \cdot 3)^3\\ &= 6^3 = 216\end{aligned}$$

8. $2^2 \cdot 5^2$
9. $4^3 \cdot 25^3$
10. $20^2 \cdot 5^2$
11. $3^4 \cdot 2^4$

110. The power of a quotient. The following illustration shows how to find the power of a quotient:

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a^3}{b^3}$$

Similarly, show that:

$$\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}; \quad \left(\frac{2x}{3y}\right)^4 = \frac{(2x)^4}{(3y)^4}$$

Show that $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$; or $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$

Express this law in words.

EXERCISES

Express the following powers as quotients:

1. $\left(\frac{xy}{2ab}\right)^4$

3. $\left(\frac{10ab}{9ac}\right)^3$

5. $\left(\frac{-2mt}{-5pq}\right)^4$

$$\left(\frac{xy}{2ab}\right)^4=\frac{(xy)^4}{(2ab)^4}=\frac{x^4y^4}{16a^4b^4}$$

2. $\left(\frac{3bc}{5x}\right)^2$

4. $\left(-\frac{3xy}{4bz}\right)^4$

6. $\left(\frac{(2x)(3y)}{(4a)(-b)}\right)^3$

Find the value of the following:

7. $\frac{42^3}{7^3}$

8. $\frac{6^5}{3^5}$

10. $\frac{81^2}{9^2}$

$$\frac{42^3}{7^3}=\left(\frac{42}{7}\right)^3=6^3$$

9. $\frac{8^4}{4^4}$

11. $\frac{90^2}{15^2}$

111. The power of a power. The following example illustrates how to find the power of a power:

$$(a^5)^3=a^5\cdot a^5\cdot a^5=a^{15}$$

Similarly, show that $(2^4)^3=2^{12}$; $(a^3)^2=a^6$

Show that $$(a^m)^n=a^{mn}=(a^n)^m$$

Express this law in words.

EXERCISES

Simplify the following:

1. $[(-2)^2]^3$

4. $(a^4)^3$

7. $(x^{a-b})^{a+b}$

2. $(-2^2)^3$

5. $(b^2)^3$

8. $(a^{x-4})^{x-1}$

3. $(x^6)^2$

6. $(a^{n+1})^2$

9. $(a^{2x})^{x-2}$

MISCELLANEOUS EXERCISES

112. Write the following in the simplest form:

1. $\left(\frac{2a^2b^3}{4x^2y}\right)^3$

2. $\left(\frac{3a^2}{2b}\right)^3 \cdot \left(\frac{3b^2}{4a}\right)^2$

3. $\frac{(x^2b^3)^5}{(xb^2)^3}$

4. $\left(-\frac{2b^3x}{5a^2y}\right)^{2n}$

5. $\left(\frac{3xy}{4z}\right)^8 \div \left(\frac{9x}{8z^2}\right)^4$

6. $\left[\left(-\frac{mn^2y^3}{pq^2z}\right)^3\right]^2$

7. $\left(\frac{a^3b^4c^n}{ab^3c^2}\right)^n$

8. $(a^2-b^2)^2$

9. $\left(\frac{a^3+b^3}{a+b}\right)^2$

Zero Exponents. Fractional and Negative Exponents

113. The symbol a^m is a brief expression for

$$a \cdot a \cdot a \ldots . \ (m \text{ times}).$$

Accordingly m must be a positive *integer*. In the following we shall find a meaning for *negative, fractional,* and *zero*-exponents.

114. Zero exponents. The law $a^m \div a^n = a^{m-n}$ has been shown to hold for positive integral values of m and n, with the understanding that m is greater than n.

If we *assume* the law to hold also for $m=n$, we have

$$\frac{a^2}{a^2}=a^{2-2}=a^0; \quad \frac{a^3}{a^3}=a^{3-3}=a^0$$

However, by dividing numerator and denominator of each fraction by the common factors a, we have

$$\frac{a^2}{a^2}=\frac{\cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a}}=1; \quad \frac{a^3}{a^3}=\frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}}=1$$

Thus we may think of a^0 as *the result obtained by dividing a power by itself,* and we may *assign to it* the value **1.**

In general, if we assume the law $a^m \div a^n = a^{m-n}$ to hold for $m=n$, we have

$$\frac{a^m}{a^m} = a^{m-m} = a^0$$

By reducing the fraction $\frac{a^m}{a^m}$ to the simplest form we have

$$\frac{a^m}{a^m} = 1$$

Hence we may *assign* to the symbol a^0 the value 1, i.e.,

$$\mathbf{a^0 = 1}$$

It should be added that a must *not* be zero. For, 0^m is 0, and $\frac{0^m}{0^m} = \frac{0}{0}$ has no meaning.

EXERCISES

Give the value of each of the following expressions: 10^0; x^0; $(-15)^0$; $(x+y)^0$; $(a-b+c)^0$.

115. Negative exponent. Assuming the law

$$\mathbf{a^m \div a^n = a^{m-n}}$$

to hold also if $m<n$, we have for $m=3$ and $n=5$:

$$\frac{a^3}{a^5} = a^{3-5} = a^{-2}$$

Since by reducing $\frac{a^3}{a^5}$ to the simplest form, we have

$$\frac{a^3}{a^5} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a} = \frac{1}{a^2}$$

we may *define* a^{-2} to mean $\frac{1}{a^2}$.

Similarly, $a^{-1} = \frac{1}{a}$, $a^{-3} = \frac{1}{a^3}$, $a^{-4} = \frac{1}{a^4}$.

In general, $$\mathbf{a^{-m} = \frac{1}{a^m}}$$

EXERCISES

Find the value of each of the following expressions:

1. $2^5 \cdot 2^0$
2. $6(a-b)^0$
3. $\dfrac{1}{3^{-4}}$
4. $8 \cdot 2^{-4}$
5. $5^3 \cdot 5^{-2}$
6. $\left(\dfrac{3}{4}\right)^{-1}$
7. 3^{-2}
8. $.125^{-1}$
9. $\left(\dfrac{2}{3}\right)^{-3}$

Change the following to identical expressions free from negative exponents and simplify:

10. $a^8 \cdot a^{-2}$
11. $(-2x^{-4})(-3x^{-1})$
12. $-4a \div a^{-3}$
13. $(2a)^{-2}b^2$
14. $\dfrac{2a^{-2}b}{3a^3b^{-4}}$
15. $\left(\dfrac{4x^2y^2}{3zx^{-2}}\right)^{-1}$
16. $\dfrac{(x^2y^{-1})^{-2}}{(a^2b^{-2})^{-2}}$
17. $(a+a^{-1})(a-a^{-1})$
18. $(x+x^{-1})^2$

Solve the following equations for x, assuming $a \neq 0$, $a \neq 1$:

19. $a^{x-2}=a^3$
20. $a^{2x+5}=a^{7-x}$
21. $(a^{x-2})^{x-1}=(a^{4-x})^{1-x}$
22. $4^{x+1}=8 \cdot 2^{x+2}$

116. Fractional exponent. Assuming the law

$$a^m \cdot a^n = a^{m+n}$$

to hold for fractional exponents, we have

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}=a;\quad a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}}=a;\quad a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} \cdot a^{\frac{1}{4}}=a;\ \text{etc.}$$

Since, $\sqrt{a} \cdot \sqrt{a}=a$; $\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a}=a$; $\sqrt[4]{a} \cdot \sqrt[4]{a} \cdot \sqrt[4]{a} \cdot \sqrt[4]{a}=a$, etc., we may define $a^{\frac{1}{2}}$ to represent the same number as $\sqrt{a}$. Similarly,

$$a^{\frac{1}{3}}=\sqrt[3]{a};\quad a^{\frac{1}{4}}=\sqrt[4]{a};\ \text{etc.}$$

In general, $$a^{\frac{1}{m}}=\sqrt[m]{a}$$

Show that $\left(a^{\frac{1}{n}}\right)^m = a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \ldots (m \text{ factors})$

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}\cdots(m \text{ terms})} = a^{\frac{m}{n}}$$

Show that $(a^m)^{\frac{1}{n}} = [a \cdot a \cdot a \ldots (m \text{ factors})]^{\frac{1}{n}}$

$$= a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \ldots (m \text{ factors})$$

$$\therefore (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}},$$

$$\text{or } \sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{\frac{m}{n}}$$

Thus a *fractional* exponent indicates a root of a power or a power of a root, the *numerator* indicating the power, and the *denominator* the root.

EXERCISES

Find the value of each of the following expressions:

1. $4^{\frac{1}{2}}$
2. $27^{\frac{1}{3}}$
3. $(-8)^{\frac{1}{3}}$
4. $(64)^{\frac{1}{3}}$
5. $(-125)^{\frac{1}{3}}$
6. $49^{-\frac{1}{2}}$
7. $\left(\frac{27}{64}\right)^{-\frac{1}{3}}$
8. $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$
9. $(.25)^{\frac{1}{2}}$
10. $(32x^{-5}b^{10})^{\frac{1}{5}}$

117. Summary of the laws of exponents and of the meaning of fractional, negative, and zero exponent. The laws and definitions given in §§ 107 to 116 are as follows:

I. $a^m \cdot a^n = a^{m+n}$

II. $\dfrac{a^m}{a^n} = a^{m-n}$

III. $(a \cdot b \cdot c)^m = a^m b^m c^m$

IV. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

V. $(a^m)^n = a^{mn}$

VI. $a^0 = 1,\ a \neq 0$

VII. $a^{-m} = \dfrac{1}{a^m}$

VIII. $a^{\frac{1}{n}} = \sqrt[n]{a}$

IX. $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

These laws have been shown to hold for all *rational* values of m and n. However, we shall find that, by enlarging our conception of powers, quite clear and definite meanings can be given to powers with *irrational* exponents § 146, and even to powers with *imaginary* exponents.

MISCELLANEOUS EXERCISES

118. Change the following to identical expressions free from negative or zero exponents. Simplify as far as possible:

1. $\left(\dfrac{x}{y}\right)^{\frac{1}{2}}\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$

2. $\left(x^{-\frac{1}{n}}\right)^{-\frac{1}{m}}$

3. $(\sqrt[m]{a})^{2m}$

4. $\left(x^{\frac{m+n}{p}}\right)^{\frac{p}{m-n}}$

5. $(ab^{-2}c^3)^{\frac{2}{3}}$

6. $\dfrac{b^{\frac{n-1}{n}}}{b}$

7. $\dfrac{x^3y^2z^{-5}}{x^{-1}yz^2}$

8. $\dfrac{a^{-\frac{1}{2}}}{a^{\frac{5}{8}}}$

9. $\left(\dfrac{a^{-1}b^{-3}c^0}{a^2b^2}\right)^2$

10. $(\sqrt[4]{x})^{\frac{2}{3}}(\sqrt[3]{x^{-4}})^{\frac{1}{4}}$

11. $\left(\dfrac{4x}{9y}\right)^{\frac{1}{2}} \div \dfrac{4x^{\frac{1}{6}}}{12y^{\frac{1}{4}}}$

12. $\dfrac{a+a^{-1}}{a-a^{-1}}$

13. $a^{-3}+b^{-3}$

14. $\dfrac{a^4}{a^{-4}+a^{-5}}$

15. $x^{-1}+2x^{-2}+3x^{-3}$

16. $\dfrac{x^{-2}+y^{-2}}{x^{-2}y^{-2}}$

17. $\left(\dfrac{a^{-1}b}{a^{\frac{1}{2}}b^{-\frac{2}{3}}}\right) \div \left(\dfrac{a^{-3}}{b^{-1}}\right)^{-\frac{1}{3}}$

18. $\left(\dfrac{a^3b^{-4}}{a^{-2}b}\right)^3 \cdot \left(\dfrac{a^{-3}b^2}{xb^{-1}}\right)^5$

Multiply as indicated:

19. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}-y^{\frac{1}{2}})$

20. $(x^{-4}+x^{-2}+x)(x-2)$

21. $(x^{-\frac{1}{3}}-y^{-\frac{2}{3}})^3$

22. $(x^{-2}+x^{-1}y^{-1}+y^{-2})(x^{-1}-y^{-1})$

23. $(a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b)(a^{\frac{1}{4}}-b^{\frac{1}{4}})$

24. $(mx^{-2}+ny^{-1}+p)^2$

Divide as indicated:

25. $(a^{\frac{5}{2}}+a^2+a^{\frac{3}{2}}+a+a^{\frac{1}{2}}+1)$ by $(a+a^{\frac{1}{2}})$

26. $(3y^{\frac{4}{3}}+6x+9x^{\frac{1}{4}}y^{\frac{1}{3}}+2x^{\frac{3}{4}}y)$ by $(y+3x^{\frac{1}{4}})$

Extract the square root of:

27. $4a^2-4ab^{\frac{1}{3}}+4ac^{-\frac{1}{2}}+b^{\frac{2}{3}}-2b^{\frac{1}{3}}c^{-\frac{1}{2}}+c^{-1}$

28. $x^{\frac{4}{3}}y^{-1}+4xy^{-\frac{1}{2}}-2x^{\frac{2}{3}}-12x^{\frac{1}{3}}y^{\frac{1}{2}}+9y$

Reduce the following fractions:

29. $\dfrac{x-y}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$; $\dfrac{x+y}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$

Exercises Taken from College Entrance Examinations

‡119. Simplify the following:

1. $(2^{\frac{1}{2}}\times 2^{\frac{2}{3}})\div 54^{-\frac{1}{3}}$ (Sheffield)

2. $xy^{\frac{2}{3}}\left(\dfrac{x^{-\frac{1}{2}}}{y^{-\frac{2}{3}}}\right)^2\div(x^{\frac{1}{4}}y^{-3})^{-\frac{2}{3}}$ (Board)

3. $2(8)^{\frac{2}{3}}-\sqrt{3}(12)^{\frac{1}{2}}-2(3)^0+(a^{\frac{1}{3}}b^{-1})^3b^3-\dfrac{1}{a^{-1}+b^{-1}}$ (Board)

4. $(ayx^{-1})^{\frac{1}{2}}\cdot(bxy^{-2})^{\frac{1}{3}}\cdot(y^2a^{-2}b^{-2})^{\frac{1}{4}}$ (Princeton)

5. $\dfrac{(4p^4q)^3}{(9p^2q^2)^4}\div\dfrac{(p^2q^5)^2}{2q^5}$ (Yale)

6. $\dfrac{x^{-1}y^0z^{-3}}{x^{-2}y^3z^2}$; $\dfrac{a^0x^{-3}y^{\frac{1}{2}}}{a^2x^{-2}y^{-\frac{3}{2}}}$ (Yale)

7. $(a^4+x^4)(a^2-x^2)^{-\frac{3}{2}}-(a^2-x^2)^{\frac{1}{2}}$

8. $\dfrac{ax(a^{-1}x-ax^{-1})}{x^{\frac{2}{3}}-a^{\frac{2}{3}}}$

9. $[x^{-4a}+x^{-2a}y^{2b}+y^{4b}]\div\left[\dfrac{1}{x^{2a}}-\dfrac{x^{-a}}{y^{-b}}+y^{2b}\right]$ (Chicago)

10. $\left(\dfrac{x^q}{x^r}\right)^{q+r}\left(\dfrac{x^r}{x^p}\right)^{r+p}\left(\dfrac{x^p}{x^q}\right)^{p+q}$ (M.I.T.)

11. Solve for x: $(x+1+x^{-1})(x-1+x^{-1})=5\frac{1}{4}$

12. Multiply: $x^{-\frac{1}{2}}y^{\frac{1}{3}}+x^{\frac{1}{4}}y^{-\frac{1}{6}}$ by $x^{\frac{1}{2}}-2y^{\frac{1}{6}}$

13. Find the square root of
$a^{-2}+9b^{\frac{4}{3}}+16c^{-\frac{1}{2}}+6a^{-1}b^{\frac{2}{3}}-8a^{-1}c^{-\frac{1}{4}}-24b^{\frac{2}{3}}c^{-\frac{1}{4}}$ (Board)

14. Find the value of $\dfrac{1+8^{-\frac{x}{3}}}{(8x)^{\frac{1}{2}}+10^{x-2}}$, when $x=2$ (Cornell)

15. Find the value of x^2+y^2 in terms of u and v when

$$x=u+v\frac{v^{-\frac{1}{3}}}{u^{-\frac{1}{3}}},\quad y=v+u\frac{u^{-\frac{1}{3}}}{v^{-\frac{1}{3}}};$$

and reduce the answer to its simplest form (Harvard)

Radicals

120. Radical. Radicand. Index. Order. An indicated root of a number, as $\sqrt{5}$, $\sqrt[3]{64}$, $\sqrt[4]{a+1}$, is a **radical.**

The number of which the root is to be taken is the **radicand.**

Name the radicands in $\sqrt{3}$, $2\sqrt[4]{x+2}$, $\sqrt[3]{a}$.

The number that indicates what root is to be taken is the **index** of the root.

Give the index in each of the following radicals: $\sqrt[4]{26}$, $\sqrt[3]{xy}$, $\sqrt{2}$, $\sqrt[n]{a}$.

The index gives the **order** of the radical.

EXERCISES

1. Recalling the meaning of the square root of a number, find the value of $\sqrt{2}\sqrt{2}$; of $\sqrt{3}\sqrt{3}$; of $\sqrt{a}\sqrt{a}$; of $(\sqrt{a})^2$.

2. Find the value of $\sqrt[3]{3}\sqrt[3]{3}\sqrt[3]{3}$; of $\sqrt[3]{4}\sqrt[3]{4}\sqrt[3]{4}$; of $\sqrt[3]{a}\sqrt[3]{a}\sqrt[3]{a}$; of $(\sqrt[3]{a})^3$.

3. What is the meaning of $\sqrt{a}$?

4. Since $\sqrt[n]{a}\equiv a^{\frac{1}{n}}$ show that $(\sqrt[n]{a})^n=a$; $\sqrt[n]{a^n}=a$.

5. Show that $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$, or $\sqrt[n]{ab}=\sqrt[n]{a}\sqrt[n]{b}$.

6. State the principles expressed in exercises 4 and 5.

7. Find the value of $\sqrt{9}$; $\sqrt[3]{64}$; $\sqrt[4]{81}$; $\sqrt[5]{32}$.

8. Multiply $(a+\sqrt{b})$ by $(a-\sqrt{b})$;
$(\sqrt{x}+\sqrt{y})$ by $(\sqrt{x}-\sqrt{y})$.

9. Reduce $\dfrac{a}{\sqrt{a}}$; $\dfrac{x+y}{\sqrt{x+y}}$; $\dfrac{m-n}{\sqrt{m}+\sqrt{n}}$.

10. Find the value of $\sqrt{3}\sqrt{27}$; $\sqrt[3]{9}\sqrt[3]{3}$; $\sqrt[3]{a^2}\sqrt[3]{a}$.

11. Find the value of $\sqrt{100-36}$; $\sqrt{16+9}$; $(4+3\sqrt{2})\ (4-3\sqrt{2})$.

Reduction of Radicals

121. Removal of a factor from the radicand. The following examples illustrate the method of removing factors from the radicand.

1. $\sqrt{8}=\sqrt{4\cdot 2}=\sqrt{4}\cdot\sqrt{2}=2\sqrt{2}$.
2. $\sqrt[3]{16x^5y^2}=\sqrt[3]{8\cdot 2\cdot x^3\cdot x^2y^2}=\sqrt[3]{8\cdot x^3}\sqrt[3]{2x^2y^2}=2x\sqrt[3]{2x^2y^2}$.
3. $\sqrt[3]{(a+b)^2(a^2-b^2)}=\sqrt[3]{(a+b)^2(a+b)(a-b)}=(a+b)\sqrt[3]{a-b}$.

Thus a radical may be reduced if the radicand contains factors to powers equal to, or greater than, the index of the radical.

EXERCISES

Reduce the following radicals to the simplest form:

1. $\sqrt{75}$

2. $\sqrt{45}$

3. $\sqrt{98}$

4. $3\sqrt[3]{40}$

5. $\sqrt[4]{1250}$

6. $\sqrt[4]{512}$

7. $\sqrt[3]{8(a+b)^6}$

8. $5\sqrt[3]{24x^6y^5z^4}$

9. $\sqrt[6]{216a^3b^6c^9}$

10. $\sqrt{a^2b^2+a^3b^4}$

11. $\sqrt{9ab^2-9b^3}$

12. $\sqrt{a^3+3a^2b+3ab^2+b^3}$

13. $\sqrt[4]{81m^5n^4}$

14. $\sqrt{3x^3+18x^2+27x}$

15. $\sqrt{(x^2-5x+6)(x^2-3x+2)}$

16. $\sqrt{a^2x^4+2abx^4+b^2x^4}$

122. Reduction of a fractional radicand to the integral form. The following examples illustrate the method of changing a fractional to an integral radicand:

1. $\sqrt{\frac{1}{3}}=\sqrt{\frac{1\cdot 3}{3\cdot 3}}=\frac{1}{3}\sqrt{3}$ 2. $\sqrt{\frac{2}{3x}}=\sqrt{\frac{2\cdot 3x}{3x\cdot 3x}}=\frac{1}{3x}\sqrt{6x}$

3. $\sqrt[3]{\frac{81}{4x^4}}=\sqrt[3]{\frac{27\cdot 3}{4x^3\cdot x}}=\frac{3}{x}\sqrt[3]{\frac{3}{4x}}=\frac{3}{x}\sqrt[3]{\frac{3\cdot 2x^2}{4x\cdot 2x^2}}=\frac{3}{2x^2}\sqrt[3]{6x^2}$

Hence, to change a fractional to an integral radicand, *reduce the radicand to the simplest form. Then multiply numerator and denominator by a number which will make the denominator a power whose exponent is the same as the index of the radical.*

EXERCISES

Reduce the following to the simplest form:

1. $\sqrt[3]{\frac{8x^4y}{27z^4}}$ **4.** $\sqrt[3]{\frac{27a}{5b^2}}$ **7.** $\sqrt[6]{\frac{4}{125a^3}}$

2. $3\sqrt{\frac{3a}{25x^2}}$ **5.** $\sqrt[4]{\frac{5a^3}{9b^3}}$ **8.** $\sqrt{\frac{a^2+c^2}{ac}}$

3. $\sqrt[3]{\frac{120}{15}}$ **6.** $\sqrt{\frac{5x^3}{3a^2y}}$ **9.** $\sqrt{\frac{a+b}{a-b}}$

123. Reducing the order of a radical. Show from the following examples how to reduce the *order* of a radical:

1. $\sqrt[6]{8}=(2^3)^{\frac{1}{6}}=2^{\frac{1}{2}}=\sqrt{2}$

2. $\sqrt[6]{x^4}=x^{\frac{4}{6}}=x^{\frac{2}{3}}=\sqrt[3]{x^2}$ 3. $\sqrt[9]{64}=(2^6)^{\frac{1}{9}}=2^{\frac{2}{3}}=\sqrt[3]{4}$

EXERCISES

Reduce to radicals of lower order:

1. $\sqrt[4]{25}$ **4.** $\sqrt[12]{64}$ **7.** $\sqrt[4]{9}$

2. $\sqrt[6]{27}$ **5.** $\sqrt[10]{32}$ **8.** $\sqrt[6]{125a^3b^3}$

3. $\sqrt[4]{a^2b^4}$ **6.** $\sqrt[6]{49}$ **9.** $\sqrt[9]{27x^6y^3}$

Addition and Subtraction of Radicals

124. Similar radicals. If radicals when reduced to the simplest form have the *same index* and the *same radicand* they are **similar.**

Show that $3\sqrt{6}$ and $5\sqrt{6}$ are similar radicals; also $\sqrt{x^3y}$ and $3\sqrt{xy}$.

If radicals are to be added or subtracted they are first reduced to the simplest form. Similar radicals are then combined.

Thus, $\sqrt{2}+\sqrt{8}+3\sqrt{50}=\sqrt{2}+2\sqrt{2}+15\sqrt{2}=18\sqrt{2}$

$$3\sqrt{\frac{3}{8}}-\sqrt{24}+\sqrt{\frac{2}{3}}=3\sqrt{\frac{6}{16}}-\sqrt{4\cdot 6}+\sqrt{\frac{6}{9}}$$

$$=\left(\frac{3}{4}-2+\frac{1}{3}\right)\sqrt{6}=\frac{11}{12}\sqrt{6}$$

$$\sqrt{4x+4y}-\sqrt{16x^3+16x^2y}=2\sqrt{x+y}-4x\sqrt{x+y}$$

$$=(2-4x)\sqrt{x+y}.$$

EXERCISES

Simplify and collect similar terms:

1. $4\sqrt[3]{2}-2\sqrt[3]{54}+\sqrt[3]{128}$
2. $3\sqrt{98}-7\sqrt{80}-3\sqrt{18}$
3. $10\sqrt{\frac{6}{5}}-\sqrt{\frac{3}{10}}+4\sqrt{\frac{15}{2}}$
4. $2\sqrt{\frac{4}{7}}+5\sqrt{\frac{9}{7}}+\sqrt{175}$
5. $\sqrt{\frac{3a}{b}}+\sqrt{\frac{3b}{a}}-\sqrt{\frac{ab}{3}}$
6. $3\sqrt{98}-25\sqrt{3}+\sqrt{108}$
7. $28^{\frac{1}{2}}+\sqrt{63}+\sqrt{112}$
8. $\sqrt[6]{9}+2\sqrt[9]{27}-2\sqrt[3]{-24}$
9. $\sqrt[4]{32a^5}+\sqrt[4]{512a}+\sqrt[4]{2a}$
10. $\sqrt{28}-\sqrt{7}+3\sqrt{175}$
11. $4\sqrt{1+a^2}-\sqrt{9+9a^2}-2\sqrt{b^2+a^2b^2}$

Multiplication of Radicals

125. Multiplication of radicals of the same order. Before multiplying, all radicals should be reduced to the simplest form:

Show that $\sqrt{6}\cdot\sqrt{7}=\sqrt{6\cdot 7}=\sqrt{42}$

$$3\sqrt[3]{xy^4c}\cdot 5\sqrt[3]{cd^5}=3y\sqrt[3]{xyc}\cdot 5d\sqrt[3]{cd^2}$$
$$=15dy\sqrt[3]{xyc^2d^2}$$

EXERCISES

Multiply as indicated:

1. $\sqrt{a^3}\cdot\sqrt{a}$
2. $\sqrt[3]{x^5}\sqrt[3]{a^2}$
3. $3\sqrt{5}\cdot\sqrt{10}\cdot 7\sqrt{35}$
4. $(\sqrt{2}-\sqrt{3})\sqrt{5}$
5. $(2\sqrt{5}-5)(3-\sqrt{5})$
6. $(\sqrt{6}+\sqrt{10})(\sqrt{3}-\sqrt{5})$

126. Multiplication of radicals of different orders. *If the radicals to be multiplied do not have the same index, they should be changed to the same order before multiplying.*

For example, $\sqrt[3]{4}\sqrt{5}=4^{\frac{1}{3}}\cdot 5^{\frac{1}{2}}=4^{\frac{2}{6}}\cdot 5^{\frac{3}{6}}=\sqrt[6]{4^2}\sqrt[6]{5^3}=\sqrt[6]{2000}$

Similarly, $4\sqrt{xy}\cdot 3\sqrt[3]{x^2y}=4\sqrt[6]{x^3y^3}\cdot 3\sqrt[6]{x^4y^2}=12\sqrt[6]{x^7y^5}$
$$=12x\sqrt[6]{xy^5}$$

EXERCISES

Multiply as indicated:

1. $\sqrt[4]{x^3}\sqrt{x}$
2. $\sqrt[3]{x^2}\sqrt{x^3}$
3. $\sqrt{2x}\sqrt[3]{3x^2}$
4. $\sqrt[3]{x^2}\sqrt{x}$
5. $\sqrt[3]{2}\sqrt[6]{4}$
6. $\sqrt[3]{9x^2}\cdot\sqrt{15x}$

Division of Radicals

127. Division by a monomial. As in multiplication, radicals are to be brought to the same index before dividing.

Thus: $\sqrt[5]{x^3}\div\sqrt{x^5}=\sqrt[5]{x^3}\div x^2\sqrt{x}=\sqrt[10]{x^6}\div x^2\sqrt[10]{x^5}=\frac{1}{x^2}\sqrt[10]{x}$

EXERCISES

Divide as indicated:

1. $\dfrac{\sqrt{12}}{\sqrt{2}}$

2. $\dfrac{\sqrt{5}}{\sqrt{3}}$

3. $\dfrac{\sqrt{xy}}{\sqrt{x}}$

4. $\dfrac{2\sqrt[3]{54}}{\sqrt{2}}$

5. $\dfrac{6\sqrt{x}}{2\sqrt[3]{x}}$

6. $\dfrac{\sqrt[3]{3}}{\sqrt[6]{9}}$

MISCELLANEOUS EXERCISES

128. Simplify the following:

1. $\sqrt{\dfrac{36a^3}{3^4b}\cdot\dfrac{3^2c^{-2}}{24a}\cdot\dfrac{9b}{8c^4}}$

2. $\left(\dfrac{x^{-3}}{y^{-\frac{2}{3}}z}\right)^{-\frac{3}{2}}\div\left(\dfrac{\sqrt{x^{-\frac{1}{2}}}\sqrt[6]{y^3}}{x^2z^{-1}}\right)^{-2}$ (Princeton)

3. $\sqrt[7]{x^2y^{12}}\left(\dfrac{1}{xy}\right)^{\frac{1}{7}}\left(\dfrac{y^2}{x^3}\right)^{-\frac{2}{7}}$ (Sheffield)

4. $(\sqrt[3]{2x}\div\sqrt{8x^3}$; $(1-x)\div(1-\sqrt[3]{x})$;

 $2^{\frac{1}{3}}\times\sqrt[6]{\dfrac{1}{3}}\times3^{\frac{1}{3}}$; $(a+b)\left(\dfrac{a-b}{a+b}\right)^{\frac{1}{2}}$

5. $\left(\dfrac{\sqrt[5]{3}}{3^{-\frac{4}{5}}}\right)^3-\sqrt{3\cdot 27^{-1}}+(-243)^{\frac{2}{5}}+\left(\dfrac{\sqrt{3^{-1}}}{2^{-1}}\right)^2$ (Chicago)

6. $-x^2(9-x^2)^{-\frac{1}{2}}+\sqrt{9-x^2}+\dfrac{3}{\sqrt{1-\left(\dfrac{x}{3}\right)^2}}$ (Sheffield)

7. $\sqrt{\dfrac{x+y}{x-y}}-\sqrt{\dfrac{x-y}{x+y}}+\dfrac{2x}{x^2-y^2}\sqrt{x^2-y^2}$ (Sheffield)

8. $\sqrt{a^3-a^2b}-\sqrt{ab^2-b^3}-\sqrt{(a+b)(a^2-b^2)}$ (Chicago)

9. $3\sqrt{\dfrac{5}{2}}+\sqrt{40}+\sqrt{\dfrac{2}{5}}-\dfrac{1}{\sqrt{10}}$ (Sheffield)

10. $3\sqrt{\dfrac{2}{5}}+2\sqrt{\dfrac{1}{10}}-4\sqrt{\dfrac{1}{40}}$ (Sheffield)

11. $\sqrt{2}\times\sqrt[3]{3}$; $2\sqrt{20}-\sqrt[4]{80}+\sqrt{\frac{4}{5}}$ (Sheffield)

12. $\sqrt{\frac{1}{7}}+\sqrt{63}+5\sqrt{7}$; $(4\sqrt[5]{7}-8\sqrt[5]{21}+6\sqrt[5]{42})\div 2\sqrt[5]{7}$

13. $\sqrt[4]{\frac{9}{625}}+6\sqrt{\frac{1}{3}}-\sqrt{12}$; $a\sqrt{\frac{x}{a}}+x\sqrt{\frac{a}{x}}$

14. Determine, without extracting roots, which one of the following is the greatest: $\sqrt[3]{10}$, $\sqrt{6}$, $\sqrt[4]{17}$.

Rationalizing the Denominator

129. Rationalizing the denominator. The process of changing a fraction with **irrational** denominator to an equivalent fraction with **rational** denominator is called **rationalizing the denominator.** By means of this process it is possible to avoid dividing by a decimal fraction when the *value* of the fraction is required.

For example, to find the value of $\frac{1}{2+\sqrt{3}}$ it would be necessary to approximate the square root of 3, to add the result to 2, and to divide the sum into 1.

However, multiplying numerator and denominator by $2-\sqrt{3}$, we have

$$\frac{1}{2+\sqrt{3}}=\frac{1(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}$$

Hence the value of $\frac{1}{2+\sqrt{3}}$ may be found easily by subtracting $\sqrt{3}$ from 2.

Moreover, by rationalizing the denominator it is possible to reduce the number of square roots required to find the value of a fraction.

For example, $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\dfrac{(\sqrt{5}-\sqrt{2})^2}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})}=\dfrac{7-2\sqrt{10}}{3}$

It is seen that the *given* fraction calls for the approximation of two square roots and division by a decimal fraction. *After rationalizing* the denominator it is necessary to extract *only one* square root and to divide by 3.

The number by which numerator and denominator are multiplied to make the denominator rational is called the **rationalizing factor** of the denominator.

Radical expressions of the forms $a+\sqrt{b}$ and $a-\sqrt{b}$; $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$ are called **conjugate radicals.**

EXERCISES

Change the following fractions to equivalent fractions with rational denominator:

1. $\dfrac{\sqrt{5}}{\sqrt{3}}$

$$\frac{\sqrt{5}}{\sqrt{3}}=\frac{\sqrt{5}\sqrt{3}}{\sqrt{3}\sqrt{3}}=\frac{\sqrt{15}}{3}$$

2. $\dfrac{3}{\sqrt{2}}$

3. $\dfrac{3+\sqrt{18}}{\sqrt{3}}$

4. $\dfrac{a}{\sqrt{a+b}}$

5. $\dfrac{\sqrt{2}}{\sqrt{3}-1}$

6. $\dfrac{\sqrt{2}}{2-\sqrt{2}}$

7. $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$

8. $\dfrac{3\sqrt{5}-4}{2\sqrt{5}+3}$

9. $\dfrac{\sqrt{2}+2\sqrt{5}}{2\sqrt{2}-3\sqrt{5}}$

10. $\dfrac{3\sqrt{3}-\sqrt{7}}{3\sqrt{2}+\sqrt{7}}$

11. $\dfrac{3\sqrt{2}-\sqrt{5}}{\sqrt{5}-6\sqrt{2}}$

12. $\dfrac{1+2\sqrt{3}}{2+\sqrt{3}+\sqrt{5}}$

13. $\dfrac{2-\sqrt{6}}{3+\sqrt{2}-\sqrt{6}}$

14. Find the value of each of the fractions in exercises 6 to 10 to three significant figures.

MISCELLANEOUS EXERCISES

‡**130.** Solve the following exercises:

1. Solve for x: $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x+2a}+\sqrt{x-2a}}=\dfrac{x}{2a}$ (Board)

2. Simplify: $\left(\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)^2+\left(\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)^2$ (Board)

3. Simplify: $\sqrt{\dfrac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}}$ (Board)

4. Solve: $\dfrac{1}{x}\left[\dfrac{1}{\dfrac{1}{x}+1}-\dfrac{1}{\dfrac{1}{x}-1}\right]+\dfrac{1+x\sqrt{2}}{1-x\sqrt{3}}=\dfrac{3x^2-2x-3}{1-x^2}$ (Yale)

5. Find the approximate value of $\dfrac{\sqrt{3}+4}{\sqrt{2}-\sqrt{6}}\div\dfrac{\sqrt{2}+2\sqrt{6}}{\sqrt{3}-2}$ to three decimal places. (M.I.T.)

6. Rationalize and find correct to two decimal places:

$$\frac{1}{2+\sqrt{5}-\sqrt{2}} \quad \text{(Yale)}$$

7. Simplify $\dfrac{\sqrt{2}+2\sqrt{3}}{\sqrt{2}-\sqrt{12}}$ and compute the value of the fraction to two decimal places. (Yale)

8. Find the value of x from the equation $5x=\sqrt{3}(1+2x)$ and express it as a fraction having a rational denominator.

9. Simplify: $2x^2\sqrt{9x^2+81}+27\sqrt{4x^2+36}$

Square Root of a Radical Expression

131. By squaring the binomial $\sqrt{a}+\sqrt{b}$ we have $a+b+2\sqrt{ab}$. Therefore $\sqrt{a}+\sqrt{b}$ is the square root of the binomial $(a+b)+2\sqrt{ab}$.

Hence it is possible to find the square root of a binomial of the form $x+a\sqrt{y}$ if it can be changed to the form $a+b+2\sqrt{ab}$.

The following examples illustrate the process:

1. Find the square root of $8+\sqrt{48}$.

$$8+\sqrt{48}=8+2\sqrt{12}=8+2\sqrt{6\cdot 2}=6+2+2\sqrt{6\cdot 2}$$

$$\therefore\ \sqrt{8+\sqrt{48}}=\sqrt{6}+\sqrt{2}$$

2. Find the square root of $38+3\sqrt{32}$.

$$38+3\sqrt{32}=38+\sqrt{9\cdot 32}=38+\sqrt{9\cdot 8\cdot 4}$$
$$=38+2\sqrt{72}$$
$$=38+2\sqrt{36\cdot 2}=36+2+2\sqrt{72}$$

$$\therefore\ \sqrt{38+3\sqrt{32}}=6+\sqrt{2}$$

EXERCISES

Find the square root of each of exercises 1 to 9:

1. $3-2\sqrt{2}$ **4.** $7+4\sqrt{3}$ **7.** $3-\sqrt{5}$

2. $6-2\sqrt{8}$ **5.** $11-3\sqrt{8}$ **8.** $7+\sqrt{48}$

3. $11-4\sqrt{7}$ **6.** $14+6\sqrt{5}$ **9.** $11-6\sqrt{2}$

10. In finding the sine of 15° by two different methods, we obtain the results $\frac{1}{4}(\sqrt{6}-\sqrt{2})$ and $\frac{1}{2}\sqrt{2-\sqrt{3}}$, respectively. Show that these results have the same value.

Irrational Equations

132. Irrational equations in the form of quadratics. By changing the form some irrational equations may be solved like quadratics.

For example, the equation

$$7x^2-5x+1-8\sqrt{7x^2-5x+1}=-15$$

may be changed to

$$a^2-8a+15=0,$$

where $$a=\sqrt{7x^2-5x+1}$$

Show that $$a_1=5,\ a_2=3$$

$$\therefore\ \sqrt{7x^2-5x+1}=5,\ \sqrt{7x^2-5x+1}=3$$

By squaring both sides of these equations, quadratic equations are found which may be solved for x.

EXERCISES

Solve the following equations:

1. $x^2-5x+2\sqrt{x^2-5x-2}=10$

Subtract 2 from both sides of the equation.

2. $6y^2-3y-2=\sqrt{2y^2-y}$

Notice that $6y^2-3y=3(2y^2-y)$

3. $x^2-3x+4+\sqrt{x^2-3x+15}=19$

4. $y^{\frac{2}{3}}-\frac{5}{8}y^{\frac{1}{3}}-1=0$

5. $4x^{-\frac{1}{6}}-3x^{-\frac{1}{3}}-1=0$

6. $x^{-\frac{4}{3}}-5x^{-\frac{2}{3}}+4=0$

7. $3y^{-\frac{3}{2}}+20y^{-\frac{3}{4}}-32=0$

8. $(a+2)^{\frac{1}{2}}-(a+2)^{\frac{1}{4}}-2=0$

9. $\sqrt[3]{7a-6}+4=4\sqrt[6]{7a-6}$

133. Irrational equations solved by reducing them to rational equations. The following examples show the method of solving irrational equations which can be reduced to rational equations.

1. Solve $\sqrt{x+13}-\sqrt{x+6}=1$, and check.

Adding $\sqrt{x+6}$ to both sides, $\sqrt{x+13}=1+\sqrt{x+6}$

Squaring, $x+13=1+2\sqrt{x+6}+x+6$

$$\therefore 3=\sqrt{x+6}$$

Squaring again, $9=x+6$

$$x=3$$

2. Solve $\sqrt{x}+\sqrt{2x+1}-\sqrt{5x+5}=0$

Isolating $\sqrt{5x+5}$, $\sqrt{x}+\sqrt{2x+1}=\sqrt{5x+5}$

Squaring, $x+2\sqrt{2x^2+x}+2x+1=5x+5$

$$2\sqrt{2x^2+x}=2x+4$$

Dividing by 2, $\sqrt{2x^2+x}=x+2$

Squaring, $2x^2+x=x^2+4x+4$

$$\therefore x_1=4,\ x_2=-1$$

The value $x=-1$ *does not satisfy the original equation,* but it satisfies the *third* equation. Hence it was brought in by the process of squaring the second equation. It is said to be an *extraneous root* of the original equation.

Example 2 shows that the results obtained by solving an irrational equation are *not always roots.* Hence it is necessary to *check* all results in the *original* equation.

EXERCISES

Solve the following equations and check the results:

1. $y+2\sqrt{y-1}-4=0$ (Sheffield)
2. $\sqrt{x+4}+\sqrt{2x-1}=6$ (Sheffield)
3. $\sqrt{7x+1}-\sqrt{3x+10}=1$ (Board)
4. $\sqrt{x+20}-\sqrt{x-1}=3$ (Board)
5. $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}$ (Princeton)
6. $\sqrt{a-x}+\sqrt{a+x}=\sqrt{2a+2b}$
7. $\sqrt{x+5}+\sqrt{2x+8}=\sqrt{7x+21}$ (Princeton)
8. $\sqrt{7x-5}+\sqrt{4x-1}=\sqrt{7x-4}+\sqrt{4x-2}$ (Harvard)
9. $\dfrac{\sqrt{5x-4}+\sqrt{5-x}}{\sqrt{5x-4}-\sqrt{5-x}}=\dfrac{2\sqrt{x}+1}{2\sqrt{x}-1}$

Apply the process of *addition* and *subtraction.*

10. $\sqrt{8x-7}-\dfrac{2x-2}{\sqrt{2x+3}}=\sqrt{2x+3}$

Clear of fractions.

11. $\sqrt{3+x}+\sqrt{x}=\dfrac{6}{\sqrt{3+x}}$

‡12. $\dfrac{2y-3}{\sqrt{y-2}}=2\sqrt{y-2}-1$ 13. $\dfrac{ax-1}{\sqrt{ax}+1}-4=\dfrac{\sqrt{ax}-1}{2}$

Reduce the first fraction to the simplest form.

‡14. $\dfrac{x-b}{\sqrt{x}+b}=\dfrac{\sqrt{x}-\sqrt{b}}{3}+2\sqrt{b}$ 15. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x-b}{\sqrt{x-b}}=\sqrt{a-b}$

‡**16.** Reduce $\sqrt{(x-4)^2+y^2}+\sqrt{(x+4)^2+y^2}=10$ to an equation free from radicals and as compact as possible. (Board)

‡**17.** Simplify the following expression as far as possible:

$$\sqrt{x^3+ax^2-a^2x-a^3}-\sqrt{x^3-3ax^2+3a^2x-a^3}-a\sqrt{4x-4a}$$

Assume that both $x-a$ and $x+a$ are positive. (Harvard)

Trigonometric Equations

134. Some trigonometric equations reduce to irrational equations.

For example, $\tan\theta+\sec\theta=3$.

Since $\sec\theta=\sqrt{1+\tan^2\theta}$, we have

$$\sqrt{1+\tan^2\theta}=3-\tan\theta$$

Squaring both sides, $1+\tan^2\theta=9-6\tan\theta+\tan^2\theta$

$$\therefore 6\tan\theta=8$$

$$\tan\theta=\frac{4}{3}$$

The value of θ may be found from a table of tangents.

EXERCISES

Solve the following equations:

1. $2\sin\theta=1+\cos\theta$

2. $\sin\theta+\cos\theta=\sqrt{2}$

Summary

135. The chapter has taught the meaning of the following terms:

base	index of a root
exponent	order of a radical
power	rationalizing a number
radical	irrational equations
similar radicals	zero exponent
conjugate radicals	negative exponent
radicand	fractional exponent

136. The following problems review the essentials of the chapter:

1. Give the meaning of each of the following:

I. $a^m \cdot a^n = a^{m+n}$

II. $\dfrac{a^m}{a^n} = a^{m-n}$

III. $(a \cdot b \cdot c)^m = a^m b^m c^n$

IV. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

V. $(a^m)^n = a^{mn}$

VI. $a^0 = 1$

VII. $a^{-m} = \dfrac{1}{a^m}$

VIII. $a^{\frac{1}{n}} = \sqrt[n]{a}$

IX. $a^{\frac{m}{n}} = \begin{cases} \sqrt[n]{a^m} \\ (\sqrt[n]{a})^m \end{cases}$

2. Explain how to solve the following:

1. To change an expression containing negative or zero exponents to an identical expression free from negative or zero exponents.

2. To remove a factor from a radicand.

3. To reduce a fractional radicand to the integral form.

4. To reduce the order of a radical.

5. To add and subtract radicals.

6. To multiply radicals of the same order, or of different orders.

7. To divide by a radical.

8. To rationalize the denominator of a fraction.

9. To find the square root of a binomial of the form $x + a\sqrt{y}$.

10. To solve irrational equations.

11. To solve trigonometric equations leading to irrational equations.

CHAPTER VII

LOGARITHMS. SLIDE RULE

Labor-saving Devices

137. Precision of measurement. When we compare a line-segment with a known segment such as an inch or a centimeter, we are *measuring* the line-segment.

To measure AB, Fig. 58, we may lay off the distance AB on squared paper, as $A'B'$.

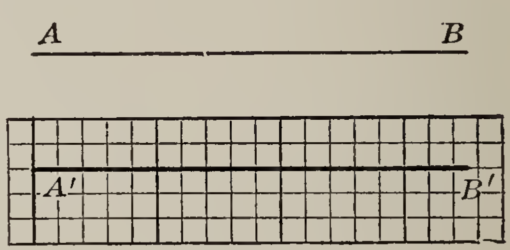


Fig. 58

Using 2 cm. as a unit we find the measure of $A'B'$ to be 1.76. The 6 being estimated, the number 1.76 does not mean that the length of AB is exactly 1.76, but rather that it is between 1.755 and 1.765. The result, 1.76, is said to be expressed to three *significant figures*, the *precision* being indicated by the *number of figures.* Three-figure accuracy may be obtained with ordinary instruments. In surveying, four-figure accuracy is usually sufficient, but with skill and good instruments five-figure accuracy is possible.

138. Abridged multiplication. In adding, subtracting, multiplying, or dividing two numbers obtained by measurement it is useless to express the result to greater accuracy than that of the *less accurate* of the original numbers. Much labor may be avoided by omitting the meaningless figures in the product or quotient. The

following example illustrates the process of *abbreviating* the *multiplication* of two numbers which are known only approximately:

Find the product of 2.4301 by 7.8043, to five significant figures.

The work may be arranged as follows:

$$
\begin{array}{r|l}
\multicolumn{2}{c}{2.430\underline{1}\times 7.804\underline{3}} \\
\hline
 & \underline{72903} \\
9 & 720\underline{4} \\
1944 & 0\underline{8} \\
17010 & \underline{7} \\
\hline
18965 & \underline{22943}
\end{array}
$$

A study of the complete process of multiplying the two numbers brings out the following facts: Since in 7.8043, the last figure, 3, is uncertain, the first partial product, 72903 is uncertain. This may be indicated by a line drawn under each figure. Similarly, the 4 in the second partial product, 97204, is uncertain, etc. Evidently the final product, 18.96522943, is accurate only to five significant figures, the last figure, 5, being uncertain. Hence we may omit in all partial products the part to the right of the vertical line.

A further simplification is obtained by writing the partial products in the reverse order, i.e., multiply 2.4301 by 7, then 2.430 by 8, 2.4 by 4, and 2 by 3.

The work may now be arranged in the following form:

$$
\begin{array}{r|l}
\multicolumn{2}{c}{2.430\underline{1}\times 7.804\underline{3}} \\
\hline
17.010 & \underline{7} \\
1.944 & 0 \\
9 & 6 \\
 & 6 \\
\hline
18.964 & \underline{9}
\end{array}
$$

EXERCISES

1. Find by abridged multiplication the following products:

$$12.13\times 119.4;\quad 14.625\times .32814;\quad .1342\times 2.16$$

2. A cubic centimeter of mercury weighs 13.596 g., approximately. What is the weight of 7.43 cubic centimeters?

3. How many square inches are contained in a square meter, if a meter is approximately 39.37 inches?

139. Abridged division. The process is illustrated on the following example:

Divide 6.384 by 1.231.

$$\begin{array}{r|l}
6.38\bar{4} & 1.23\bar{1} = 5.189 \\
6.15\bar{5} & \\
\hline
229 & \\
12\bar{3} & \\
\hline
106 & \\
9\bar{6} & \\
\hline
10 & \\
\bar{9} & \\
\hline
1 &
\end{array}$$

Placing the divisor to the right of the dividend and marking the uncertain figures by a line, we find that 1231 is contained in 6384 five times, leaving the remainder 229. We now cut off the last figure in the divisor and find that 123 is contained once in 229, leaving the remainder 106. The process of cutting off a figure from the divisor is kept up until the whole divisor is used. Hence, the quotient is 5.189, the 9 being uncertain.

EXERCISES

1. Find the following quotients:

$$63.4 \div 26.8;\quad 86.423 \div 18.25$$

2. The lunar month has 29.531 days. How many lunar months are there in a year which is equal to 365.24 days?

140. Use of logarithms. By the use of *logarithms* the operations of multiplication and division may be reduced to addition and subtraction.

When *only one* multiplication or division is to be made it can be performed quickly by using the abridged pro-

cesses given in §§ 138 and 139. The use of logarithms is especially valuable where a *series* of operations is involved.

The meaning of logarithms and the theory of computation by logarithms will be discussed fully in this chapter, §§145 to 160.

141. Use of the slide rule. Products, quotients, powers, and roots of numbers may be found *mechanically* by an instrument called the *slide rule.* A knowledge of logarithms is necessary to understand the principles on which the slide rule is constructed. A full discussion of these principles and of the use of the slide rule is found in §§ 163 to 166.

142. Use of tables. The student is familiar with tables of roots and powers. They may be used to save time and to avoid unnecessary labor.

Logarithms

143. Table of exponents. The table, Fig. 59, gives the first 25 powers of 2. By means of this table it is

$2=2^1$	$1{,}024=2^{10}$	$262{,}144=2^{18}$
$4=2^2$	$2{,}048=2^{11}$	$524{,}288=2^{19}$
$8=2^3$	$4{,}096=2^{12}$	$1{,}048{,}576=2^{20}$
$16=2^4$	$8{,}192=2^{13}$	$2{,}097{,}152=2^{21}$
$32=2^5$	$16{,}384=2^{14}$	$4{,}194{,}304=2^{22}$
$64=2^6$	$32{,}768=2^{15}$	$8{,}388{,}608=2^{23}$
$128=2^7$	$65{,}536=2^{16}$	$16{,}777{,}216=2^{24}$
$256=2^8$	$131{,}072=2^{17}$	$33{,}554{,}432=2^{25}$
$512=2^9$		

Fig. 59

possible to reduce multiplication and division of numbers to addition and subtraction respectively. This follows from the theorem $a^m \cdot a^n = a^{m+n}$.

For example, let it be required to find the product $512 \times 16{,}384$.

The table gives: $512 \times 16{,}384 = 2^9 \times 2^{14} = 2^{23} = 8{,}388{,}608$.

Thus by adding the exponents 9 and 14 we are able to locate the required product.

Similarly to find the quotient

$$524{,}288 \div 8{,}192,$$

we find from the table that

$$524{,}288 \div 8{,}192 = 2^{19} \div 2^{13} = 2^6 = 64.$$

EXERCISES

Find the value of each of the following:

1. 16×512
2. $2{,}048 \times 128$
3. $65{,}536 \times 256$
4. $4{,}194{,}304 \div 131{,}072$
5. $8{,}192 \div 256$
6. $131{,}072 \div 16{,}384$
7. $\dfrac{128 \times 16 \times 16{,}384}{256 \times 32 \times 1{,}024}$

Evidently a more complete table of exponents would be very useful in performing multiplications and divisions.

The following examples show how the table may be used to find *powers* and *roots:*

1. Find the square of 2,048.

From the table, $2{,}048 = 2^{11}$.
Therefore, $2{,}048^2 = (2^{11})^2 = 2^{22} = 4{,}194{,}304$.

2. Find the square root of 1,048,576.

From the table $1{,}048{,}576 = 2^{20}$.
Therefore, $\sqrt{1{,}048{,}576} = \sqrt{2^{20}} = (2^{20})^{\frac{1}{2}} = 2^{10} = 1{,}024$.

The *exponents* in the table, Fig. 59, are called the *logarithms* of the left members of the corresponding equations. Thus, if 2 is used as a base, the logarithm

JOHN NAPIER

JOHN NAPIER, BARON OF MERCHISTON

JOHN NAPIER, a wealthy Scotch baron, made political and religious controversy the main business of his life, but his pet amusement was the study of mathematics and science. He was born in 1550 and died in 1617. The stupendous labors in calculating of some of his contemporaries impressed him with the desirability of devising some way of shortening multiplications and divisions. Rheticus, with forty helpers, had spent years calculating the trigonometrical tables published in 1596 and 1613; Vieta seems to have enjoyed calculations requiring many days of hard labor; Ludolph von Ceulen (1539–1610) gave most of his life to calculating π to 35 decimal places; and Cataldi (1548–1626) gave years of hard labor to numerical calculating.

Napier devised a set of rods, known as "Napier's bones," containing sets of products in convenient form for facilitating multiplying.

His *virgulae*, another invention, were to aid in the extraction of square and cube roots. He discovered certain trigonometrical formulas, known as Napier's analogies, and stated his "rule of circular parts," a mnemonic aid to recalling the laws of right spherical triangles. His chief service to science was his invention of logarithms, which after suitable modification by Briggs (1561–1631), became the powerful aid to calculation that we employ today. His great work on logarithms, entitled *Rabdologia*, was published in 1617.

[See Ball, pp. 235–36, also *Encyclopaedia Britannica.*]

of 128 is 7. This is expressed briefly in symbols by the equation

$$\log_2 128 = 7,$$

read *the logarithm of 128 to the base 2 is 7.*

Express by means of equations the logarithm, to the base 2, of the following numbers: 16, 256, 2,048, 16,384.

144. Using 3 as base, we have the following table of exponents:

$3 = 3^1$	$243 = 3^5$	$19{,}683 = 3^9$
$9 = 3^2$	$729 = 3^6$	$59{,}049 = 3^{10}$
$27 = 3^3$	$2{,}187 = 3^7$	$177{,}147 = 3^{11}$
$81 = 3^4$	$6{,}561 = 3^8$	$531{,}441 = 3^{12}$

EXERCISE

Express by means of equations the logarithms, to the base 3, of the following numbers: 243, 2,187, 19,683.

145. Logarithm.* *The* ***logarithm*** *of a number, N, to the base a, is the* ***exponent*** *to which a must be raised to give a power that is equal to N.*

Thus, if $a^x = N$, then $\log_a N = x$. These two equations are equivalent.

* Tables of logarithms were first published by John Napier, a Scotch baron, in 1614. Jost Bürgi (1552–1632) had calculated and used extensively tables of logarithms before 1617, but did not publish his tables until 1620.

Henry Briggs (1561–1631) introduced the modern idea of logarithms to the base 10, and it was mainly through his influence that logarithms rapidly came into use all over Europe. Kepler introduced them in Germany about 1629, Cavalieri in Italy in 1624, and Edmund Wingate in France in 1626. Briggs also introduced the method of long division now commonly used in arithmetic. See Ball, *Short Account of the History of Mathematics*, pp. 235–37, and Tropfke, *Geschichte der Elementar-Mathematik*, Band II, S. 145–55.

EXERCISES

1. Using 10 as base find the logarithms of 10, 100, 1,000, 10,000. Express each result in the form of an equation.

2. Find the logarithm of 1 to the base 1, 2, 3..., a. Express each result in the form of an equation.

3. Find the logarithm to the base 2 of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$.

4. Find the logarithm of a number using the same number as base.

Common Logarithms

146. Common logarithms. Logarithms to the base 10 are called **common logarithms,** or Briggs' logarithms. The base 10 is generally not written, it being *understood* that 10 is the base unless another base is indicated. Thus, $\log x$ means $\log_{10}x$.

The table of exponents, § 150, contains only numbers that are exact powers of 10. Hence the values of the logarithms of these numbers can be given exactly. The logarithm of a number which is not an *exact* power of 10 is written as a decimal fraction.

Thus the logarithm of 56.23, to five decimal places, is 1.74997, since $56.23 = 10^{1.74997}$, approximately.

147. Characteristic. Mantissa. The *integral* part of a logarithm is the **characteristic,** the *fractional* part the **mantissa** of the logarithm.

148. Graphical representation of the logarithmic function. We may use the exponential equation $10^y = x$ to find corresponding values of x and y, satisfying the equation $y = \log x$. The table below gives some of these values.

y	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$
x	1	$\sqrt{10}=3.16$	$\sqrt[4]{10}=\sqrt{3.16}=1.78$	$1.78^3=5.62$	1.33

y	1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{3}{4}$
x	10	$\frac{1}{\sqrt{10}}=\frac{1}{3.16}=.316$	$\frac{1}{1.78}=.562$	.749	.178

Plotting these pairs of values we obtain the graph of the equation $y = \log x$, Fig. 60. By means of this graph

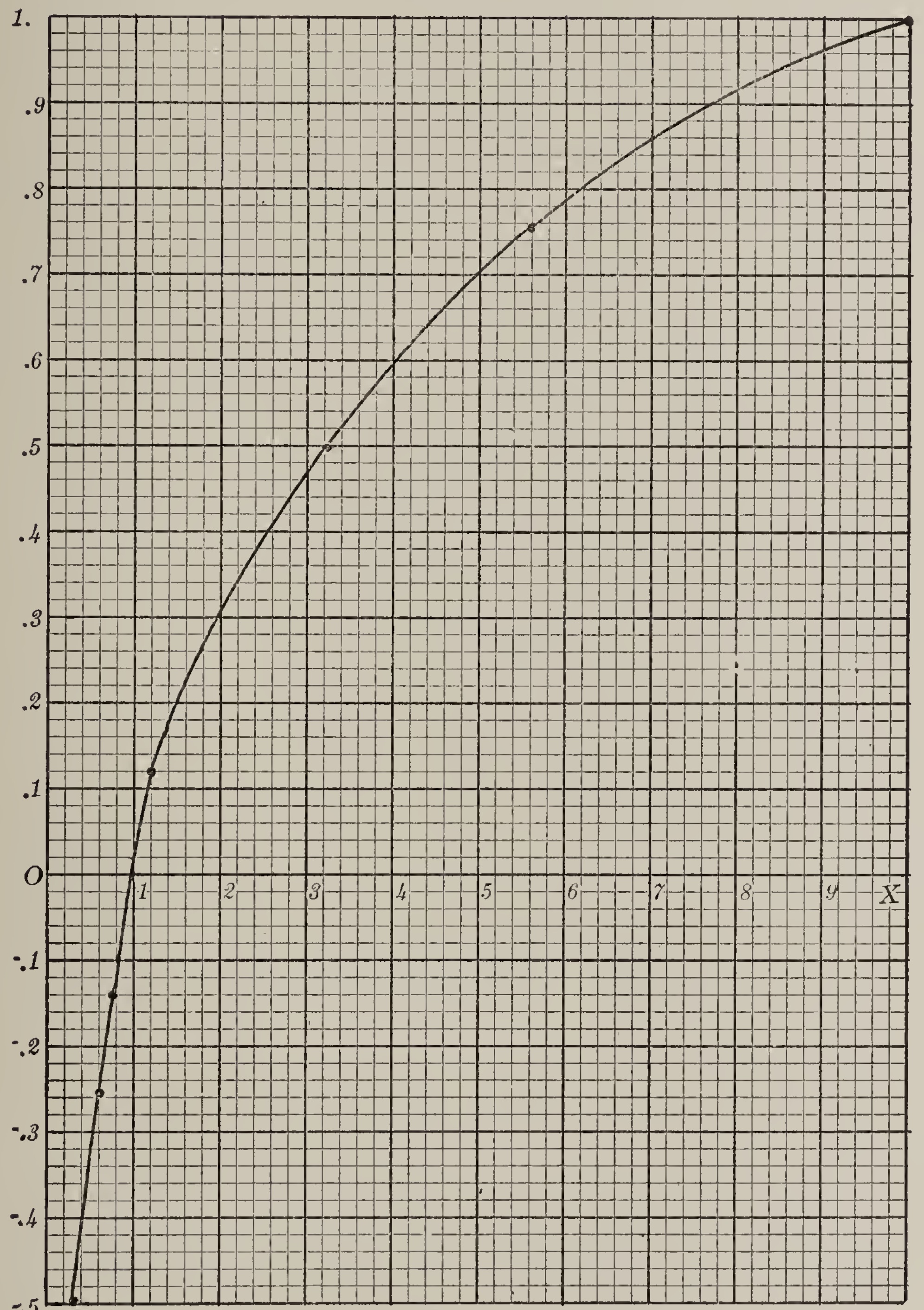


Fig. 60

it is possible to find the approximate value of the logarithm of a given number.

From the graph find the logarithm of each of the following numbers: $\frac{1}{2}$, 1.5, 4, 5, 7.4, 8.8.

A study of the graph shows the following:

1. Log x is negative for $x<1$, and increases numerically without bound as x approaches 0.

2. As x increases indefinitely, log x also increases without bound.

3. There are no logarithms of negative numbers.

Laws of physics and of other sciences frequently are of the form of logarithmic or exponential equations. Fig. 61 represents a weight, w, suspended by a rope wound a number of times about a wooden beam and kept from falling by a tension, t. The relation between w and t is given by the equation $w=te^{mx}$, m being a constant depending upon the friction between the beam and the rope, x being the number of times the rope is wound around the beam, and e being equal to 2.718, approximately.

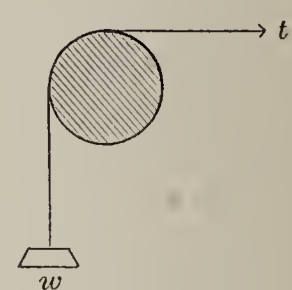


FIG. 61

EXERCISES

1. Graph the equation $y=\log_2 x$ and discuss the properties of the function $\log_2 x$.

2. Why can neither 1 nor 0 be used as base for a system of logarithms?

149. Table of logarithms. *Logarithms* may be found from special tables. From the same tables we can find the *number* corresponding to a given logarithm. Only mantissas are given, the characteristic being determined by a simple rule, § 150. If the given quantities are measured accurately to three significant figures, a four-place table will avoid inaccuracies which might affect

the results. A five-place table will give results correct to four significant figures, etc. If angles are measured to the nearest minute only, four-place tables are sufficiently accurate. If angles are measured to the nearest second, five-place tables should be used.

150. Determination of the characteristic.* The characteristic of the logarithm of a number is determined by a rule which may be derived from the following table:

From $10^5 = 100{,}000$,	follows that $\log 100{,}000 = 5$
From $10^4 = 10{,}000$,	follows that $\log 10{,}000 = 4$
From $10^3 = 1{,}000$,	follows that $\log 1{,}000 = 3$
From $10^2 = 100$,	follows that $\log 100 = 2$
From $10^1 = 10$,	follows that $\log 10 = 1$
From $10^0 = 1$,	follows that $\log 1 = 0$
From $10^{-1} = .1$,	follows that $\log .1 = -1$
From $10^{-2} = .01$,	follows that $\log .01 = -2$
From $10^{-3} = .001$,	follows that $\log .001 = -3$, etc.

Hence the characteristic of the logarithm of any number between 10,000 and 100,000 is 4; between 1,000 and 10,000 is 3; between 100 and 1,000 is 2; between 10 and 100 is 1; between 0 and 10 is 0; between .1 and 0 is -1, etc.

* The name "logarithm" is due to Napier (1614) (Tropfke, Band II, S. 176).

Cotes (1652–1716), of Cambridge, England, first speaks of a *system of logarithms* (Tropfke, Band II, S. 177). Briggs in 1624 introduced the word *characteristic,* and Wallis in 1685 introduced the term *mantissa,* in its modern mathematical sense (*ibid.*).

The phrase *base of a system of logarithms* has been current since Euler's time (1707–83). Trigonometries containing tables of trigonometric functions have been published since Vieta's time (1540–1603). See Cajori's, or Ball's, history of mathematics on the topic.

The table on p. 137 shows the following:

The characteristic of a number greater than 1 is one less than the number of digits to the left of the decimal point.

The characteristic of a number less than 1 is negative and one greater numerically than the number of zeros between the decimal point and the first significant figure of the number.

EXERCISE

What is the characteristic of each of the following numbers: 32; 8; 2,468; .8; .0021?

151. By means of a table of logarithms we find that

$$\log 7124 = 3.85272$$

It follows that $7124 = 10^{3.85272}$

Dividing both sides of this equation successively by 10, we have the following equations:

$$\begin{array}{llll}
712.4 = 10^{2.85272} & \therefore \log 712.4 & = 2.85272 \\
71.24 = 10^{1.85272} & \therefore \log 71.24 & = 1.85272 \\
7.124 = 10^{.85272} & \therefore \log 7.124 & = 0.85272 \\
.7124 = 10^{.85272-1} & \therefore \log .7124 & = 0.85272-1 \\
.07124 = 10^{.85272-2} & \therefore \log .07124 & = 0.85272-2, \text{ etc.}
\end{array}$$

This shows that a change in the position of the decimal point of a number changes the characteristic of the logarithm, but leaves the mantissa the same. Thus *all numbers having the same significant figures in the same order have the same mantissa.*

It is customary to replace the negative characteristics -1, -2, -3, etc., by $9-10$, $8-10$, $7-10$, etc., respectively.

For example,

$\log .7124 = -1 + 0.85272$ is written $9.85272-10$,

and $\log .07124 = -2 + 0.85272$ is written $8.85272-10$.

The Table of Logarithms

152. The arrangement of a table of logarithms. The general arrangement of a table of logarithms will be understood from an inspection of part of a table given below:

N	0	1	2	3	4	5	6	7	8	9
....	......	....	....	....	....	....	....	....	....	....
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96047	052	057	061	066	071	076	080	085	089
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
....	......	....	....	....	....	....	....	....	....	....
....	......	....	....	....	....	....	....	....	....	....

P.P.	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

FIG. 62

In the first column, N, we find the first three figures of the number whose logarithm is to be found. The fourth figure of the number is in the first line. The first two figures of the mantissa are in the second column, the last three in the column headed by the fourth figure of the number whose logarithm is required. An asterisk [*] before a mantissa indicates that the *first two* figures of the mantissa are to be found in the *line below* the asterisk. If a number has only 3 figures, or less, the mantissa is taken from the column headed 0. If a number has more than four figures the mantissa is found by a process of interpolation or by means of the table of proportional parts given in the last column.

153. To find the logarithm of a number. The examples on p. 140 illustrate the use of the table in finding logarithms.

1. Find the logarithm of 912.3.
   The characteristic is 2. Why?
   In the column headed N find the first three digits, 912.
   In the same line and in the column headed 3 find the mantissa, 96014, the asterisk indicating that the first two digits are 96.

$$\therefore \log 912.3 = 2.96014$$

2. Find the logarithm of 9126.3.
   The characteristic is 3. Why?
   In the column headed N find the first three digits, 912.
   In the same line and in the column headed 6 find the mantissa 96028.

$$\therefore \log 9126.0 = 3.96028$$

Similarly, $\log 9127.0 = 3.96033$

This shows that an increase of 1 in the number produces in the mantissa an increase of .00005, i.e., an increase of 5 in the fifth decimal place. It will be assumed that for equal increases in the number the corresponding increases in the logarithm are nearly equal. For the error involved is very small for sufficiently large numbers and may be neglected.

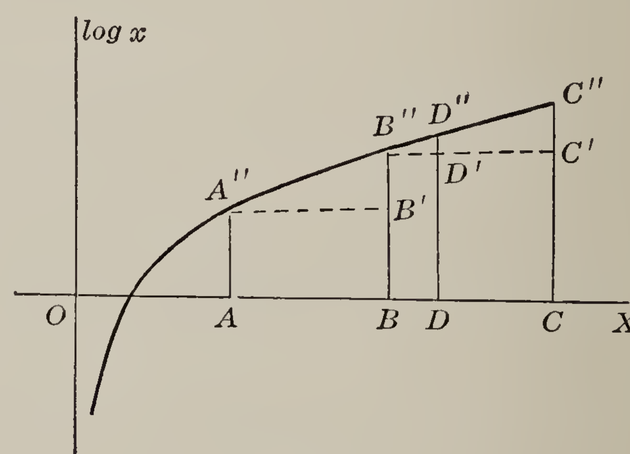


FIG. 63

To show the meaning of this assumption graphically, let OA, OB, OD, etc., Fig. 63, represent values of x, and let AA'', BB'', DD'', etc., represent the corresponding values of log x. We are assuming that $B'B'' = C'C''$, approximately, if $AB = BC$.

Moreover, for $BD = \frac{3}{10}\ BC$ we assume $D'D'' = \frac{3}{10}\ C'C''$. Since 9126.3 is $\frac{3}{10}$ of the difference between 9126 and 9127, we may find its logarithm by adding $\frac{3}{10}$ of the difference between log 9126 and log 9127 to the logarithm of 9126.

$$\begin{aligned}\therefore \log 9126.3 &= \log 9126 + \tfrac{3}{10}(\log 9127 - \log 9126)\\ &= 3.96028 + \tfrac{3}{10}(.00005)\\ &= 3.96028 + .000015\\ &= 3.960295\end{aligned}$$

Since all logarithms in the table are given only to five decimal places, the 5 in the sixth place should be omitted.

$$\therefore \log 9126.3 = 3.96029$$

The *tables of proportional parts*, headed P.P., Fig. 62, greatly facilitate the preceding computation. Thus, for a difference of 5 in the fifth places of two successive logarithms, we find $\frac{3}{10}$ of $5 = 1.5$ in the *third* line of the table of proportional parts which is headed 5.

EXERCISES

1. Using the table, Fig. 62, find the logarithm of each of the following numbers: 91.457; 0.91014; 0.0009152.

2. Using a table of logarithms find the logarithm of each of the following numbers:

1. 9; 27; 342; 875; 964.
2. 2,028; 4,516; 9,237.
3. 75,823; 64,003; 0.85992; 0.0041357.

154. To find the number corresponding to a given logarithm.

Given $\log N = 8.96120-10$. To find N.

The mantissa of $\log N$ is found to lie between the mantissas 96118 and 96123, corresponding to the numbers 9145 and 9146, respectively.

$$\begin{aligned}\text{Thus, mantissa of } \log 9145 &= 96118\\ \text{mantissa of } \log N &= 96120\\ \text{mantissa of } \log 9146 &= 96123\end{aligned}$$

It is seen that an increase of 5 in the mantissa produces an increase of 1 in the number. From the assumption that equal increases in the number produce equal increases in the logarithm, it follows that an increase of 2 in the mantissa produces an

increase in the number equal to $\frac{2}{5}$ of 1, or .4. Therefore the figures of N are 91454 and $N=0.091454$.

More conveniently, the fifth digit of N may be found by means of the table of proportional parts as follows:

Compute the tabular difference of the mantissas between which log N lies. This is found to be 5. Then compute the difference between the given mantissa of log N and the next smaller mantissa in the table. This is found to be 2. In the second column of the table of proportional parts headed 5 find the number nearest to 2. The number in the same line and in the first column is the fifth digit of N.

EXERCISE

Find the number corresponding to each of the following logarithms: 0.35021; 1.43276; 7.58142−10.

Properties of Logarithms

155. The use of logarithms in shortening computation depends upon several theorems which are proved below.

156. Logarithm of a product. Let M and N be two positive numbers whose product is to be found.

Let $\log_a M=m$ and $\log_a N=n$

then $M=a^m$ and $N=a^n$ Why?

$\therefore$ $MN=a^{m+n}$ Why?

$\therefore$ $\log_a (MN)=m+n$ Why?

$\therefore$ $\boldsymbol{\log_a (MN)=\log_a M+\log_a N}$ Why?

This equation expresses the following theorem: *The logarithm of a product is equal to the sum of the logarithms of the factors.*

For example, $\log (15\times 27)=\log 15+\log 27$.

EXERCISES

1. Show that $\log (MNR)=\log M+\log N+\log R$.

2. Given $\log 5=.6990$ and $\log 7=.8451$, find log 35.

157. Logarithm of a quotient. To find the logarithm of the quotient $\frac{M}{N}$, let

$$\log_a M = m \text{ and } \log_a N = n$$

Then $$M = a^m \text{ and } N = a^n$$

$$\therefore \quad \frac{M}{N} = a^{m-n} \qquad \text{Why?}$$

$$\therefore \quad \log_a \left(\frac{M}{N}\right) = m - n \qquad \text{Why?}$$

$$\therefore \quad \log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$$

Hence *the logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For example, $\log \frac{8}{3} = \log 8 - \log 3$

EXERCISE

Find $\log \frac{7}{5}$; $\log \frac{9}{2}$; $\log \frac{2812}{47}$.

158. Logarithm of a power. To find the logarithm of M^p, let

$$\log_a M = m$$

Then $$M = a^m$$

$$\therefore \quad M^p = (a^m)^p = a^{mp} \qquad \text{Why?}$$

$$\therefore \quad \log_a (M^p) = mp,$$

or $$\log_a (M^p) = p \log_a M$$

Hence *the logarithm of a power is equal to the exponent multiplied by the logarithm of the base.*

EXERCISE

Show that $\log (7^4) = 4 \log 7$.

159. Logarithm of a root. If $p = \frac{1}{n}$ show that the equation $\log_a (M^p) = p \log_a M$ takes the form

$$\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M,$$

i.e., *the logarithm of the n^{th} root of a number M is equal to the logarithm of M, divide by n.*

‡**160. Change of base.** Let p and q be the logarithms of N in two systems to the bases a and b, respectively.

Then $\log_a N = p$ and $\log_b N = q$

or $N = a^p$ and $N = b^q$

$$\therefore \quad q = \log_b N = \log_b (a^p) = p \log_b a$$

$$\therefore \quad \log_b N = \log_a N \cdot \log_b a$$

$$\therefore \quad \log_a N = \frac{\log_b N}{\log_b a}$$

Thus, knowing the logarithm of N in the system to the base b, we can find the logarithm of N in a system to another base a by dividing the known logarithm by $\log_b a$.

EXERCISES

1. Find $\log_3 10$; $\log_5 10$.

2. Find the value of $\log_2 \frac{1}{32} - \log \sqrt{.10}$.

3. Find the value of $\log_2 3 + \log_3 2$.

MISCELLANEOUS EXERCISES

161. Find the value of each of the expressions below. The outlines in the first three exercises suggest similar arrangements for the other exercises.

1. 254×12.26

$$\begin{aligned} \log 254 &= \\ \log 12.26 &= \\ \hline \text{adding,} \quad \log N &= \\ \therefore N &= \end{aligned}$$

2. $\dfrac{12{,}483 \times .0452}{8{,}423}$

$$
\begin{array}{lr}
 & \log 12{,}483 = \\
 & \log\ .0452 = \\
\hline
\text{adding,} & = \\
 & \log 8{,}423 = \\
\hline
\text{subtracting,} & \log N = \quad * \\
 & N =
\end{array}
$$

3. $\sqrt[3]{\dfrac{49\sqrt{352}}{86{,}420}}$

$$\log N = \frac{\log 49 + \frac{1}{2}\log 352 - \log 86{,}420}{3}$$

$$
\begin{array}{lr}
 & \log 49 = \\
 & \frac{1}{2}\log 352 = \\
\hline
\text{adding,} & = \\
 & \log 86{,}420 = \\
\hline
\text{subtracting,} & = \\
\text{dividing by 3} & \log N = \\
 & N =
\end{array}
$$

4. $\dfrac{23.40 \times .8625}{.00459 \times 6.3804}$

5. $\sqrt[3]{92}$; $\sqrt{183}$

6. $\sqrt[11]{\left(\dfrac{25.7}{286}\right)^2}$

7. $\sqrt[3]{\dfrac{0.0436}{3.187}}$

8. $\left(\dfrac{1}{\sqrt{3}}\right)^5$

9. $\sqrt{\dfrac{0.6712}{5.327}}$

10. $\dfrac{(-2582)^2 \times (.05805)}{2587 \times (-316)}$

Find the numerical value by logarithms, then prefix the proper sign.

11. $\log_4 64 - \log_3 9 + \log_2 1$ (Yale)

12. $\left(\dfrac{3.1416 \times 0.0321^2}{0.0241}\right)^{-0.32}$ (Harvard)

* When a logarithm is to be subtracted from a smaller one, 10 is both added to and subtracted from the minuend. For example, the form of the logarithm 2.34778 is changed to 12.34778–10.

13. $\dfrac{(\sqrt{278.2}\times 2.578)^3}{\sqrt[3]{.00231}\times\sqrt{76.19}}$

14. $\dfrac{3.416\times\sqrt{25.9}\,\sqrt[3]{-0.046}}{2\sqrt[3]{\frac{4}{3}}}$ (Board)

‡**15.** A number N has 17 significant figures to the left of the decimal point. What is the characteristic of $\log N$? of $\log(\log N)$? How long can this process of finding successive logarithms be kept up? (Harvard.)

‡**16.** Find by logarithms the first three figures of the number $2^{61}-1$. How many figures will this number contain? (Harvard.)

17. Given $\log 2=0.30103$, $\log 3=0.47712$. Find $\log 12$; $\log\frac{4}{3}$; $\log\frac{9}{8}$; $\log\sqrt{6}$.

Exponential Equations

162. Exponential equations. Equations in which the unknown occurs in the exponents are **exponential equations.** The following example illustrates the method of solving exponential equations by logarithms:

Solve the equation $5^x=354$

Taking the logarithm of both members,

$$\log 5^x=\log 354,$$

or

$$x\log 5=\log 354$$

$$x=\frac{\log 354}{\log 5}=\frac{2.5490}{0.6990}, \text{ etc.}$$

EXERCISES

Solve the following equations:

1. $3^y=226$

2. $2^t=437$

3. $10^y=2.71828$

4. $(3.142)^w=2.718$

5. $3^{12-2x}=243$

6. $7^{x+3}=5$

The Slide Rule

163. Description of the slide rule. The slide rule is an instrument for determining mechanically products, quotients, powers, and roots. It consists of two pieces of rule, Fig. 65, capable of sliding by each other.

Taking as unit the length A–B on the rule, Fig. 64, we may mark off, beginning from one end, the logarithms

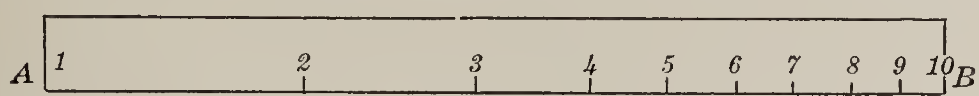


Fig. 64

of numbers from 1 to 10, or from 10 to 1,000, or from 100 to 1,000.

Thus,

$A1 = \log\ 1 = 0.$	$A6 = \log\ 6 = .78$
$A2 = \log\ 2 = .30$	$A7 = \log\ 7 = .85$
$A3 = \log\ 3 = .48$	$A8 = \log\ 8 = .90$
$A4 = \log\ 4 = .60$	$A9 = \log\ 9 = .95$
$A5 = \log\ 5 = .70$	$A10 = \log\ 10 = 1.00$

In general, the *logarithm* of a number is the *distance* from A to that number.

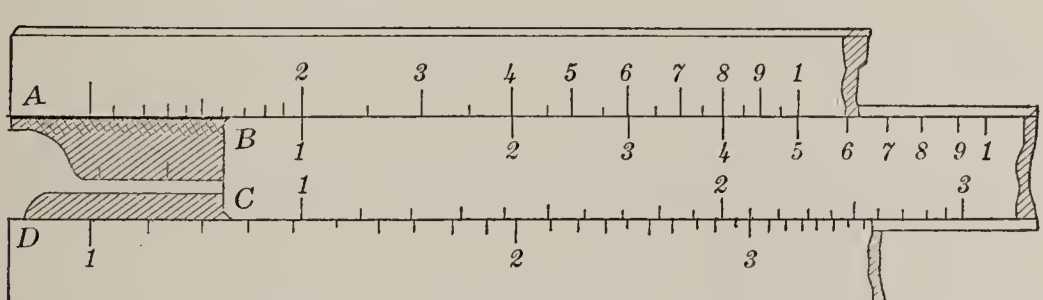


Fig. 65

The arrangement of the slide rule, Fig. 65, makes it possible to find the *sum* or *difference* of logarithms even more rapidly than with the ordinary table of logarithms.

For example, to find the *sum of two logarithms,* as $\log 2 + \log 3$, place scale B, Fig. 65, in such a way that the

division marked 1, on scale B, falls directly under the division marked 2, on scale A.

Then division 3, on scale B, falls on division 6, on scale A, and the distance $A6$, or log 6, is equal to log 2+log 3.

It is evident that this process is practically the same as that of finding the product 2×3 from a table of logarithms, which is as follows:

Let $\qquad N=2\times 3.$ Required to find N.

From the table, $\log 2=0.3010$

and $\qquad \log 3=0.4771$

adding, $\qquad \log N=0.7781$

From the table, $\qquad N=6.$

To find the *difference between two logarithms*, as log 6 − log 3, place division 3, on scale B, directly below division 6, on scale A. Then division 1 on scale B falls directly below division 2 on scale A. Hence the distance $A2$, or log 2, is equal to log 6−log 3.

Compare this process with that of finding the quotient $\frac{6}{3}$ by logarithms.

The two preceding examples show how scales A and B may be used to multiply and divide numbers.

164. The Mannheim slide rule. The *Mannheim rule*, Figs. 65 and 66, has four scales, denoted A, B, C, and D.

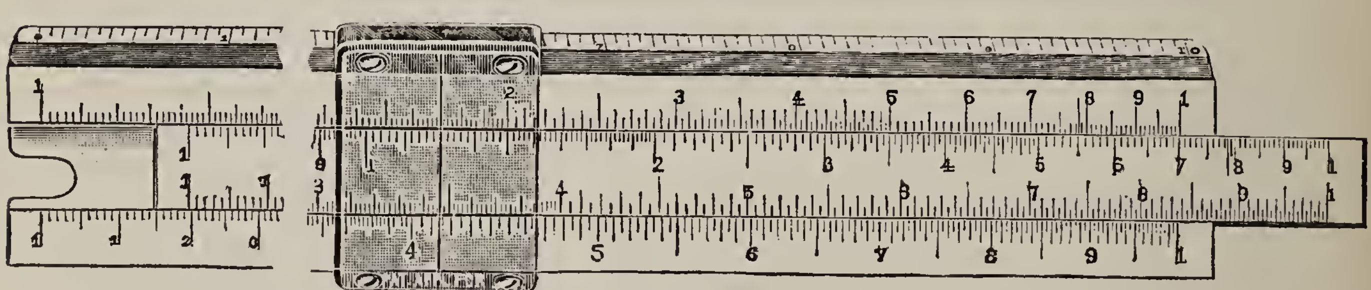


Fig. 66

Scales C and D are laid off to a scale twice as large as that of A and B. Hence the logarithm of a number on scales C or D is represented by a segment twice as large as the

segment representing the logarithm of the same number on scale A. For example, log 2 on scale $D = 2$ log 2, or log 4, on scale A; log 3 on scale $D =$ log 9 on scale A, etc. It follows that a number on scale A is the *square* of the number vertically below on scale D, and that a number on scale D is the *square root* of the number vertically above on scale A. Therefore scales A and D may be used to find the squares and the square roots of numbers.

The Mannheim rule gives results to three significant figures which is sufficiently accurate for ordinary use.

FIG. 67

If greater accuracy is required, *Thacher's slide rule*, Fig. 67, is used. This rule gives results to four or five significant figures.

The distance from 1 to 2 on scales C and D, on the Mannheim rule, is divided into 10 parts and each of these is again divided into 10 parts. These subdivisions make it possible to read off between 1 and 2 numbers containing from 2 to 4 digits, as 1.2, 1.34, 1.526, the 6 in the last number being *estimated* by the eye. Since the mantissas are the same for all numbers having the same digits in the same order, we can give the left index the value 10, 100, or 1,000. Thus, if the value of the initial 1 be 100, the divisions between 1 and 2 will be 101, 102 . . . . 110, 111 . . . . 120 . . . . 199; the division between 2 and 3 will be 200, 210, 220 . . . . ; etc.

On account of these subdivisions scales C and D, when used to multiply and divide, give more accurate results than scales A and B.

165. The use of the slide rule. The following examples illustrate the use of the rule:

1. *Multiplication.* To find the product 4×3.

The directions are given in Fig. 68: *The index 1 of scale C is put over one factor on scale D. The product is then found on scale D under the other factor on scale C.*

C	Put 1*	under 3
D	over 4	find $12 = 4 \times 3$

FIG. 68

2. *Division.* To divide 18 by 3.

Put the divisor on C over the dividend on D. Find the quotient on D under 1 on C, Fig. 69.

C	Put 3	under 1
D	over 18	find $6 = 18 \div 3$

FIG. 69

3. *The product of several factors.* To find the product of *several* factors the *runner*, r, is used.

Follow the directions in Fig. 70 to find the product $8 \times 6 \times 5 \times 2$.

C	Put 1	r to 6	1 to r	r to 5	1 to r	under 2
D	over 8					find $480 = 8 \times 6 \times 5 \times 2$

FIG. 70

4. *Reduction of fractions.* To reduce $\dfrac{26 \times 16.8 \times 35 \times 18}{42 \times 15 \times 91 \times 1.2}$.

Follow the directions given in Fig. 71.

C	Put 42	r to 168	15 to r	r to 35	91 to r	r to 1	12 to r	below 18
D	over 26							find 4 = result

FIG. 71

5. *Squares.* Find the value of 12^2.

To find the square of a number place the runner on the number, on scale D, Fig. 72.

The square is found directly above, on scale A.

A	Find $144 = 12^2$
D	Put r on 12

FIG. 72

* The *right* index 1 is used here.

6. *Square root.* To find the square root of a number proceed as follows:

Place the runner on the given number on scale A. The square root of the number is directly below, on scale D.

166. Trigonometrical computations. On the reverse side of the slide three scales are found. Scales S and T are the scales of angles. Scale A gives the sines of the angles in scale S, and scale D the tangents of the angles in scale T. The third scale gives the logarithms of the numbers on scale D. By means of these scales it is possible to find such products as $a \sin x$, or $a \tan x$.

The preceding rules exemplify most of the important applications of the slide rule. There are various makes of rules, and makers generally furnish with each rule a pamphlet giving complete instructions as to its use.

Summary

167. The chapter has taught the meaning of the following terms:

precision of measurement	table logarithms
abridged multiplication	common logarithms
abridged division	characteristic
logarithm	mantissa
slide rule	exponential equation

168. The following problems review the essential parts of the chapter:

1. Explain the processes of abridged multiplication and division.

2. Discuss the uses of logarithms and the slide rule as labor-saving devices in numerical calculations.

3. Give a discussion of the graph of the logarithmic function.

4. State the rule for determining the characteristic of a logarithm.

5. Explain (1) how to find the logarithm of a number by means of the tables; (2) how to find the number corresponding to a given logarithm.

6. State and prove the theorems regarding the properties of logarithms used in finding products, quotients, powers, and roots.

7. Explain the use of logarithms in the solution of exponential equations.

CHAPTER VIII

LOGARITHMS OF THE TRIGONOMETRIC FUNCTIONS. SOLUTION OF TRIANGLES

Use of the Table of Logarithmic Functions

169. Logarithms of trigonometric functions. In chapter vii logarithms were used to calculate expressions involving products, quotients, powers, and roots. When logarithms are to be used to find the value of an expression involving trigonometric functions the values of the functions could be looked up in a table of trigonometric functions and the logarithms of these values could then be found in a table of logarithms of numbers. To save labor the logarithms of the sines, cosines, tangents, and cotangents of angles between 0° and 90° are given in a *special table.* The logarithms of secants and cosecants are rarely used and may be obtained from the logarithms of the cosines and sines, respectively.

170. Arrangement of the table. Since the values of the sine, cosine, and tangent of angles between 0° and 45°, and of the cotangent of angles between 45° and 90° are less than 1, their logarithms will have negative characteristics. To avoid negative characteristics the form 9–10, 8–10, 7–10, etc., is used in place of -1, -2, -3, etc. The -10 is, however, omitted from the table. Hence, to have the true value of the logarithm, 10 must be subtracted from the *logarithm found in the first, second, and fourth columns.*

When the angle is less than 45°, the number of degrees is indicated at the *top* of the page and the number of

minutes is given in the *left-hand* column. When the angle is more than 45° and less than 90°, the number of degrees is indicated at the *bottom* of the page and the *right-hand* column gives the number of minutes.

171. To find the value of the logarithm of a function of a given angle. The following examples illustrate the method.

1. Find the value of log tan 52°50′12″.

The mantissa of log tan 52°50′ = 12026
The mantissa of log tan 52°51′ = 12052

∴ The tabular difference for 60″ = 26

The difference for $12'' = \frac{12}{60} \times 26 = 5.2$

This difference may be obtained quickly by means of the *table of proportional parts* as follows:

Changing 12″ to minutes, $12'' = \left(\frac{12}{60}\right)' = .2'$. This means that the required number is in the *second* line of the table headed 26.

$$\therefore \log \tan 52°50'12'' = 0.12026 + 5.2$$
$$= 0.12031$$

2. Find the value of log cot 48°25′38″.

$$\log \cot 48°25' = 9.94808 - 10$$
$$\log \cot 48°26' = 9.94783 - 10$$

Since the tabular difference is equal to 25, and since $38'' = \left(\frac{38}{60}\right)' = .63'$, we find in the sixth line of the table headed 25 the number 15.0. This means that .6 of 25 is 15. Similarly we find that .03 of 25 is .75. Hence, .63 of 25 is 15.7, or 16 units of the fifth-decimal place.

Since the cosine-function *decreases* as the angle increases, we must *subtract* 16 from the logarithm of cot 48° 25′ to get the logarithm of 48°25′38.″

$$\therefore \log \cot 48°25'38'' = 9.94808 - 10 - 16$$
$$= 9.94792 - 10$$

EXERCISES

Find the value of the following logarithms:

1. log sin 71°23′41″	**5.** log tan 27°25′10″
2. log cos 41°15′35″	**6.** log sin 41°57′36″
3. log tan 39°47′36″	**7.** log tan 37°36′.5
4. log sin 65°58′24″	**8.** log tan 23°13′.3

172. To find the angle corresponding to a given logarithmic trigonometric function. The following examples illustrate the method.

1. Given log sin $A = 9.98357 - 10$. Find A.

We find that the mantissa lies between the mantissa of log sin 74°20′ and log sin 74°21′, that the tabular difference is 3, and that the difference between the mantissa of the given logarithm and that of log sin 74°20′ is 1.

In the table of proportional parts headed 3 we find .9 nearest in value to 1. Hence we may write $1 = .9 + .1$. In the first column and in the same line with .9 we find 3.

Similarly the number nearest to .1 in the table of proportional parts is .09 and the corresponding number in the first column is .03.

$$\therefore A = 74°20'.33 = 47°20'21''$$

2. Given log cos $A = 9.85981 - 10$. Find A.

The table shows that the mantissa lies between the mantissas of log cos 43°36′ and log cos 43°37′.

The tabular difference is 12.

The difference between the mantissa of the given logarithm and that of log cos 43°36′ is 3.

Hence in the table of proportional parts headed 12 in the second column we look for the number nearest to 3. This is either 2.4 or 3.6.

Let $3 = 2.4 + .6$.

The corresponding numbers in the first columns are 2 and .05

$$\therefore A = 43°36'.25$$

EXERCISES

Find the value of A in each of the following equations:

1. $\log \sin A = 9.97527 - 10$
2. $\log \sin A = 8.73997 - 10$
3. $\log \cot A = 9.40146 - 10$
4. $\log \tan A = 0.25936$
5. $\log \cos A = 9.94749 - 10$
6. $\log \sin A = 9.42443 - 10$

Use of Logarithms in the Solution of Right Triangles

173. Solution of triangles. *To solve a triangle* is to find the values of some of the sides and angles by means of the given sides and angles. In the course of the second year right triangles were solved by use of the *natural values* of the trigonometric functions. We are now able to carry on by *logarithms* all multiplications and divisions involved in the solution.

174. Formulas. The **relations between the sides and angles of a right triangle,** Fig. 73, are expressed in the following formulas

$$\begin{aligned} a &= c \sin A & b &= c \sin B \\ a &= c \cos B & b &= c \cos A \\ a &= b \tan A & b &= a \tan B \\ a &= b \cot B & b &= a \cot A \end{aligned}$$

$$c^2 = a^2 + b^2$$

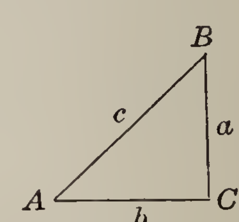


FIG. 73

The equation $c^2 = a^2 + b^2$ is usually taken in the form

$$a = \sqrt{c^2 - b^2} = \sqrt{(c+b)(c-b)}$$

The **area of the right triangle** is given by the formulas

$$S = \frac{ab}{2} = \frac{b}{2}\sqrt{(c+b)(c-b)} = \frac{b^2}{2 \tan B} = \frac{c^2}{2} \sin B \cos B$$

The preceding formulas are all adapted to logarithmic computation.

Taking the logarithm of both sides of the equations they take the forms: $\log a = \log c + \log \sin A$; $\log b = \log c + \log \sin B$, etc.

To determine an angle, the tangent or cotangent should be used because these functions change more rapidly than do the sine- and cosine-functions.

To determine a side, it is best to use the sine- or the cosine-function of the given angle.

175. Arrangement of the solution of a right triangle. The following example illustrates the plan to be followed in solving the right triangle:

Case I.—Given the sides of the right angle. To find the angles and the hypotenuse.

Let $a = 418$ and $b = 325$, Fig. 74.

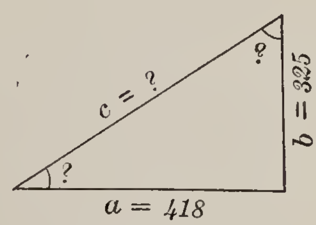


FIG. 74

(*a*) Draw a figure marking the given and required parts.

(*b*) The formulas to be used in the solution are: $\tan B = \frac{b}{a}$; $c = \frac{b}{\sin B}$; $A = 90° - B$.

The equation $a = \sqrt{(c+b)(c-b)}$ is to be used as a check.

(*c*) Make a detailed outline of the computation to be made, thus:

$$\begin{array}{rrr}
\log b = & \log b = & \text{Check: } \log (c+b) = \\
\log a = & \log \sin B = & \log (c-b) = \\
\hline
\log \tan B = & \log c = & \log a^2 = 2 \log a = \\
B = & c = & \log a = \\
A = 90° - B = & & \text{Compare this with} \\
& & \log a, \text{ found above.}
\end{array}$$

(*d*) Carry out the computation according to the plan in (*c*).

Case II.—Given the hypotenuse, c, and one of the sides, b, of the right angle.

The formulas to be used are: $\sin B = \cos A = \frac{b}{c}$; $a = \frac{b}{\tan B}$; $a = \sqrt{(c+b)(c-b)}$.

Case III.—Given one angle, B, and one of the sides, b.

Use the formulas $A=90°-B$; $a=\frac{b}{\tan B}$; $c=\frac{b}{\sin B}$; $a=\sqrt{(c+b)(c-b)}$.

Case IV.—Given one angle, B, and the hypotenuse, c.

Use the formulas $A=90°-B$; $b=c \sin B$; $a=c \cos B$; $a=\sqrt{(c+b)(c-b)}$.

EXERCISES

By means of logarithms solve the following right triangles:

1. $c=25$, $a=22$

2. $c=35.145$, $A=25°24'30''$

3. $a=316.5$, $c=521.2$

4. $B=23°9'$, $b=75.48$

5. $c=369.27$, $a=235.64$

6. $a=194.5$, $b=233.5$

7. $b=547.5$, $B=32°15'24''$

8. $c=672.4$, $B=35°16'25''$

9. $a=\ 3.414$, $b=2875$

10. $a=617.57$, $c=729.59$

Solve the following problems:

11. In order to determine the width of a river, a surveyor measured a distance of 100 ft. between two points A and B on one bank. A tree stood at a point C on the opposite bank. The angle ABC was found to be $63°40'$ and the angle BAC to be $55°35$? Calculate the width of the river. (Yale.)

12. The base of a certain triangle is 3,248 ft., and the base angles are $46°15'[=46.°25]$ and $100°37'[=100.°62]$. Find the altitude.

Sketch the figure (roughly) to scale, and see whether your result is reasonable. (Harvard.)

13. A and B are two points on opposite banks of a river 1,000 ft. apart, and P is the top of the mast of a ship directly between them. The angle of elevation of P from A is $14.°33$ $(14°20')$ and from B the angle of elevation is $8.°17$ $(8°10')$. How high is the mast? (Harvard.)

14. The shadow of a tower standing on a horizontal plane is observed to be 100 ft. longer when the sun's altitude is 30° than

when the altitude is 45°? What is the height of the tower? Do not use tables, but express the result in terms of radicals. (Yale.)

15. A circle of radius 5 subtends an angle of 20° at a point A, and M and N are the points of contact of tangents drawn from A. Find the perpendicular distance from M to AN. (Harvard.)

16. The value of the smallest division on the outer rim of a graduated circle is $30'[=0°.50]$, and the distance between the successive graduations, measured along a chord, is 0.02 inch. What is the radius of the circle? (Harvard.)

17. Each of two ships A and B, 415 yd. apart, measures the horizontal angle subtended by a cliff and the other ship; the angles are 48°17′ and 110°10′ respectively. If the angle of elevation of the cliff from A is 15°24′ what is the height of the cliff? (Board.)

18. At the top of an observation tower which is 200 ft. high and whose base is at sea-level, the angles of depression of two ships are observed to be 30°32′ and 18°40′. At the bottom of the tower the angle subtended by the line joining the two ships is found to be 40°21′. What is the distance between the ships to the nearest foot? (Board.)

19. A man who is walking on a horizontal plane toward a tower observes that at a certain point the elevation of the top of the tower is 10° and after going 50 yd. nearer to the tower the elevation is 15°. Find the height of the tower. (Princeton.)

20. The diameter of the moon is 2,164 mi. long. Find the distance from the earth to the moon if its apparent diameter subtends an angle 31′.1.

176. Isosceles triangle. The perpendicular from the vertex to the base divides the **isosceles** triangle, Fig. 75, into two congruent right triangles. Since a right triangle is determined by two parts it follows that two independent parts must be given to solve the isosceles triangle.

The following equations are used in the solution:

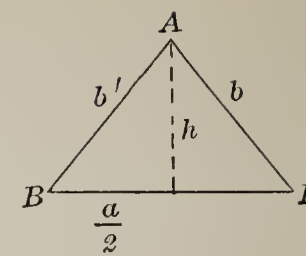


FIG. 75

$$B+\frac{A}{2}=90°$$

$$a=2b' \sin \frac{A}{2}=2b' \cos B$$

$$h=\sqrt{\left(b'+\frac{a}{2}\right)\left(b'-\frac{a}{2}\right)}=\frac{a}{2} \tan B=b' \sin B$$

$$\text{Area}=\frac{ah}{2}$$

177. Regular polygon. Lines drawn from the center of a regular polygon to the vertices divide the polygon into congruent isosceles triangles. Denoting the side by a, the radius of the inscribed or circumscribed circle by r, and the number of sides by n, we have

$\frac{a}{2}=r \sin \frac{360}{2n}$, or $a=2r \sin \frac{180}{n}$ for the inscribed polygon

and $\frac{a}{2}=r \tan \frac{360}{2n}$, or $a=2r \tan \frac{180}{n}$ for the circumscribed polygon.

The *area* in both cases is one-half the perimeter multiplied by the apothem.

Relations between the Sides and Angles of Oblique Triangles

178. By means of certain relations between the sides and angles of *oblique* triangles it will be possible to compute from certain given parts the remaining parts of a triangle. These relations are stated in the form of three laws, called the *law of sines*, the *law of cosines*, and the *law of tangents*.

179. The law of sines. 1. Let h be the length of the perpendicular from C to AB in triangle ABC, Fig. 76.

Show that $\sin A = \frac{h}{b}$ and $h = b \sin A$.

Show that $\sin B = \frac{h}{a}$ and $h = a \sin B$.

$\therefore a \sin B = b \sin A.$

It follows that $\frac{a}{\sin A} = \frac{b}{\sin B}$.

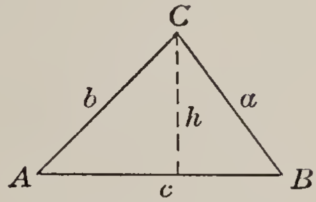


Fig. 76

By drawing a perpendicular from A to BC, we obtain in a similar way

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This equation is known as the **law of sines.** It may be expressed in words as follows: *The sides of a triangle are proportional to the sides of the opposite angles.*

In the *obtuse* triangle ABC, Fig. 77,

$$\sin A = \frac{h}{b} \text{ and } h = b \sin A$$

$$\sin x = \sin (180 - B) = \sin B = \frac{h}{a} \text{ and } h = a \sin B$$

$\therefore \quad b \sin A = a \sin B$

and $\quad \frac{a}{\sin A} = \frac{b}{\sin B}$

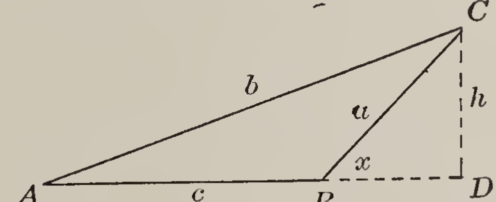


Fig. 77

It will be seen in §§ 182, 187, 189 how the law of sines is used in the solution of triangles.

180. Diameter of the circumscribed circle. The constant ratio $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ has an interesting geo-

metrical meaning. If a circle is circumscribed about $\triangle ABC$, Fig. 78, it follows that $\angle A = \angle D$.

$\therefore \quad \sin A = \sin D = \frac{a}{d}$, d denoting the diameter.

$$\therefore \quad d = \frac{a}{\sin A}.$$

Thus *the constant ratio of the side of a triangle to the sine of the opposite angle is equal to the diameter of the circumscribed circle.*

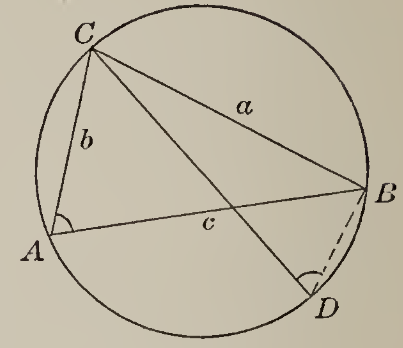


FIG. 78

181. The law of cosines. 1. Let $\angle A$, Fig. 79, be *acute.*

Then $$a^2 = b^2 + c^2 - 2cb'$$

(The square of the side opposite the acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon it.)

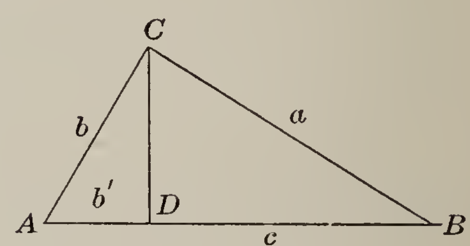


FIG. 79

Since $$b' = b \cos A,$$
it follows that $$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A.}$$
This means that *the square of a side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of these two sides and the cosine of the included angle.*

This theorem is the **law of cosines.**

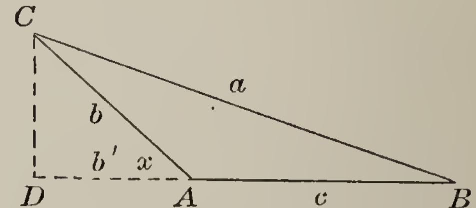


FIG. 80

2. If $\angle A$ is *obtuse*, Fig. 80,
$$a^2 = b^2 + c^2 + 2cb'$$

Since $b' = b \cos x = b \cos (180 - B) = -b \cos B$, it follows that $a^2 = b^2 + c^2 - 2bc \cos B$.

Thus the *same* equation holds for *acute* and *obtuse* angles A.

Similarly, we find

$$b^2 = c^2 + a^2 - 2ca \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Show that the theorem of Pythagoras is a special case of the law of cosines.

The following simple device makes it unnecessary to memorize *each* of these three equations.

Imagine the letters a, b, c and A, B, C placed on a circle, Fig. 81. Following the direction indicated by the arrows we pass from A to B, then to C, and again to A. By changing, in this order, the letters in the first equation above, we deduce the second equation.

Similarly the third equation may be deduced from the second. One formula is said to be obtained from the other by *cyclic substitution.*

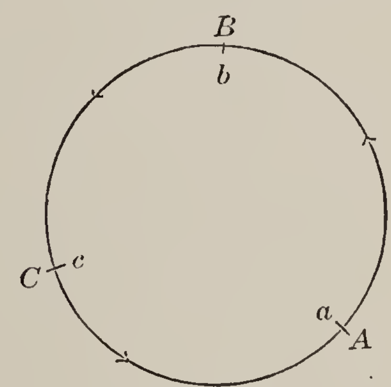


Fig. 81

182. The laws of sines and cosines are sufficient to solve oblique triangles. For, if *two angles* and *one side* are known, the equation $A+B+C=180°$ determines the third angle and the law of sines the other two sides.

If *two sides*, a and b, and the *angle*, A, *opposite one* of them are known, the third side, c, is found by solving the equation $a^2 = b^2 + c^2 - 2bc \cos A$ for c. The other angles are then found by means of the law of sines.

If *two sides* and the *included angle* are known, the law of cosines gives the third side and the law of sines the other angles.

If *three sides* are known, the law of cosines gives the angles.

However, the law of cosines is not adapted to the use of logarithms because it involves *terms* and not *factors*. The computation by means of the cosine law without logarithms is likely to be tedious for numbers containing 3 or 4 figures. Hence we shall now obtain formulas that are adapted to logarithmic computation.

183. The law of tangents. Let ABC, Fig. 82, be any triangle.

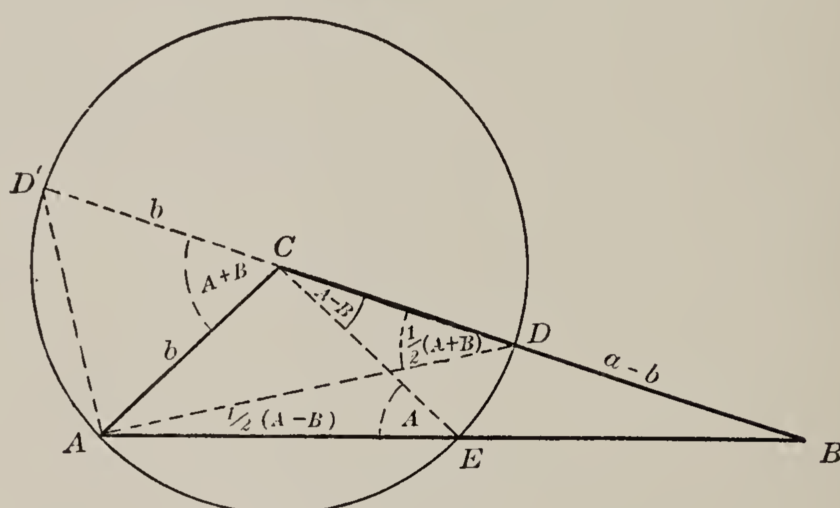


FIG. 82

With C as center and the shorter of the sides passing through C as radius, draw a circle cutting CB and AB in D and E, respectively.

Extend BC meeting the circle at D'

Draw CE, AD, and AD'

Then $\angle D'CA = A+B$. $\therefore D'DA = \frac{1}{2}(A+B)$ Why?

$A = \angle CEA = \angle ECB + B \therefore \angle ECB = A - B$ Why?

$\therefore \angle DAE = \frac{1}{2}(A-B)$

$\angle D'AD = 90°$

Applying the law of sines to $\triangle ADB$,

$$\frac{DB}{AB} = \frac{\sin DAB}{\sin ADB}, \text{ or } \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin [180° - \frac{1}{2}(A+B)]}$$

$$= \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

Hence,
$$\frac{a-b}{c}=\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tag{1}$$

Applying the law of sines to $\triangle AD'B$,

$$\frac{D'B}{AB}=\frac{\sin D'AB}{\sin AD'B},$$

$$\text{or } \frac{a+b}{c}=\frac{\sin [D'AD+\frac{1}{2}(A-B)]}{\sin AD'B}$$

$$=\frac{\sin [90°+\frac{1}{2}(A-B)]}{\sin [90°-\frac{1}{2}(A+B)]}$$

$$=\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}$$

$$\therefore \quad \frac{a+b}{c}=\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tag{2}$$

Dividing equation (2) by equation (1),

$$\frac{\dfrac{a+b}{c}}{\dfrac{a-b}{c}}=\frac{\dfrac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}}{\dfrac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}}, \text{ or } \frac{a+b}{a-b}=\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \cdot \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)}$$

$$\therefore \quad \frac{a+b}{a-b}=\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \tag{3}$$

This is called the **law of tangents.**

The formula may also be written in the form

$$\frac{b+a}{b-a}=\frac{\tan \frac{1}{2}(B+A)}{\tan \frac{1}{2}(B-A)}$$

which is to be used if $b>a$.

By cyclic substitution two similar formulas are obtained, involving b and c, and c and a, respectively.

Since

$$\frac{A+B}{2}=\frac{180}{2}-\frac{C}{2}=90°-\frac{C}{2},$$

it follows that

$$\tan \tfrac{1}{2}(A+B)=\cot \frac{C}{2}$$

∴ By substitution,

$$\frac{a+b}{a-b}=\frac{\cot \frac{C}{2}}{\tan \frac{1}{2}(A-B)}$$

Solving for $\tan \frac{1}{2}(A-B)$,

$$\tan \frac{1}{2}(A-B)=\frac{a-b}{a+b} \cot \frac{C}{2}$$

a form of the tangent law to be used when a, b, and C are known.

184. Mollweide's equations. By substituting $90°-\frac{C}{2}$ for $\frac{A+B}{2}$ in equations (1) and (2), §183,

$$\frac{a-b}{c}=\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}, \quad \frac{a+b}{c}=\frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}$$

These formulas are called **Mollweide's equations.*** They are especially useful for checking.

* Tropfke (Band II, S. 241–42) says these equations were published in 1808 by the astronomer Mollweide (1774–1825), and on account of their high applicability rapidly found wide acceptance under his name. The naming is, however, false to history, for at least the second one was known a hundred years earlier. Newton proved it in substance in his *Arithmetica universalis* of 1707.

Both equations were derived as independent trigonometric theorems by F. W. De Oppel in his *Analysis triangulorum* of 1746.

185. Tangents of half the angles of a triangle. Radius of the inscribed circle. Let O be the point of intersection of the bisectors of the angles of a triangle and let r be the length of the radius of the inscribed circle. Let x, y, and z denote the lengths of the tangents from A, B, and C, respectively.

Then the perimeter $=2x+2y+2z$.

Denoting the perimeter by $2s$ and dividing by 2,

$$s=x+y+z$$

Since,

$$a= \quad y+z$$

$$\therefore \quad s-a=x$$

Similarly, $\quad s-b=y; \ s-c=z$

From $\triangle AOD$, $\tan\frac{A}{2}=\frac{r}{x}$

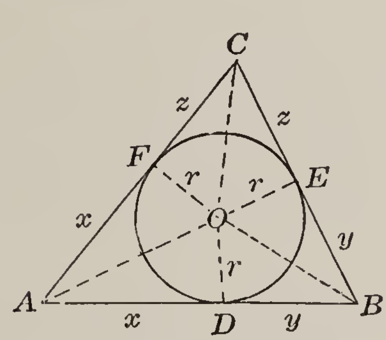


FIG. 83

$$\therefore \quad \tan\frac{A}{2}=\frac{r}{s-a}.$$

Similarly,

$$\tan\frac{B}{2}=\frac{r}{s-b}; \quad \tan\frac{C}{2}=\frac{r}{s-c}$$

From plane geometry it is known that the area F of $\triangle ABC$ is given by the formulas

$$F=\sqrt{s(s-a)(s-b)(s-c)}$$

Before Mollweide's rediscovery they are found also in Thomas Simpson's *Trigonometrie* (1765, 2d ed.) and in Mauduit's *Principles of Astronomy* of 1765.

Oppel derived the equations from the law of tangents, Simpson gave a geometrical proof, and Mauduit was content with merely applying Napier's Analogies to plane triangles. Mollweide derived the equations from the law of sines. His real service was to draw effective attention to the great usefulness of these equations in astronomy.

and $$F=\frac{r}{2}(a+b+c)=rs$$

$$\therefore \quad r=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

Hence, $$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Solution of Oblique Triangles

186. An oblique triangle can be *constructed* if three of the six parts are known, at least one of these being a side. Hence in solving oblique triangles we shall consider the following four cases.

187. Case I. Given one side and two angles. The following example illustrates the method of solution:

Given: $\begin{cases} A=49°38'30'' \\ B=70°21'15'' \\ b=229.38 \end{cases}$ *Required:* C, a, and c

Formulas: $\begin{cases} C=180°-(A+B) \\ a=\dfrac{b \sin A}{\sin B} \\ c=\dfrac{b \sin C}{\sin B} \end{cases}$

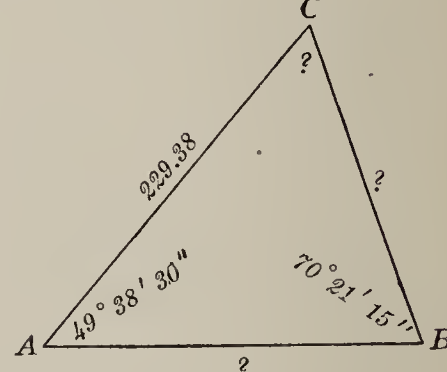


FIG. 84

Solution: $180°=179°59'60''$
$A+B=119°59'45''$

$\therefore \quad C=60° \quad 15''$

$\log b=2.36055$	$\log b=2.36055$
$\log \sin A=9.88198-10$	$\log \sin C=9.93757-10$
Adding, $12.24253-10$	Adding, $12.29812-10$
$\log \sin B=9.97396-10$	$\log \sin B=9.97396-10$
Subtracting,	Subtracting,
$\log a=2.26857$	$\log c=2.32416$
$\therefore a=185.59$	$\therefore c=210.94$

Check: $B-A=20°42'45''$	$\log c=2.32416$
$\frac{1}{2}(B-A)=10°21'22''$	$\log\sin\frac{1}{2}(B-A)=9.25473-10$
$\frac{1}{2}C=30°\ \ 7''$	Adding, $11.57889-10$
$b-a=43.79$	$\log\cos\frac{1}{2}C=9.93752-10$
$\log(b-a)=1.64137$	Subtracting, $\log(b-a)=1.64137$

Since five-place tables are used, the results are computed only to 5 significant figures, the last figure being uncertain, and angles are taken only to the nearest second. Hence fractions of a second are discarded.

188. Cologarithm. The principle $\log\frac{1}{N}=\log 1-\log N$ may be used to *avoid* the *subtractions* in the solution above.

Since $\log 1=0$, we have $\log\frac{1}{N}=0-\log N$ $=(10-\log N)-10$. The logarithm of $\frac{1}{N}$ is called the **cologarithm** of N. When *more than one* addition and subtraction is involved the use of the cologarithm has a real advantage as it is very easy to subtract *mentally* a logarithm from 10. If cologarithms are used, the solution in § 187 is arranged as follows:

$\log b=2.36055$	$\log b=2.36055$
$\log\sin A=9.88198-10$	$\log\sin C=9.93757-10$
$\text{colog}\sin B=0.02604$	$\text{colog}\sin B=0.02604$
adding, $\log a=2.26857$	adding, $\log c=2.32416$

EXERCISES

Solve the following triangles and check the results:

1. $a=29.73$, $A=52°36'$, $B=67°40'$

2. $a=788$, $C=72°12'35''$, $B=55°43'18''$

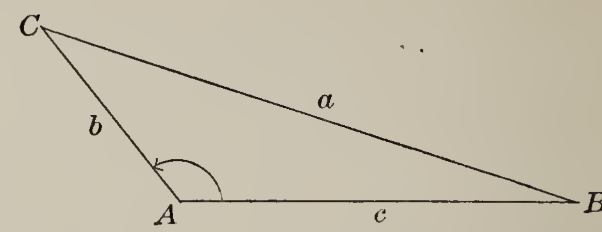
3. $c=3795$, $A=18°53'22''$, $B=81°12'5''$

4. $b=37$, $A=115°36'24''$, $B=27°18'10''$

5. $c=913.45$, $A=64°56'18''$, $B=47°29'11''$

6. $c=327.85$, $A=40°31'42''$, $B=110°52'54''$

189. Case II. Given two sides and the angle opposite one of them.



It is known from geometry that it is not always possible to construct a triangle with these given parts.

1. We will consider first the case where the angle A is *obtuse*. Then the side opposite A is the greatest side of the triangle, and one and only one triangle can be constructed satisfying the given conditions.

2. If $\angle A$ is a *right* angle, *one* triangle can be constructed. The solution of the right triangle has been discussed in § 175.

3. If $\angle A$ is acute, various possibilities may arise:

(1) If $a<h$, the length of the perpendicular from D to AB, the circle will not meet AB, and there is *no* triangle satisfying the given conditions, i.e., *no* solution of the problem exists, Fig. 85.

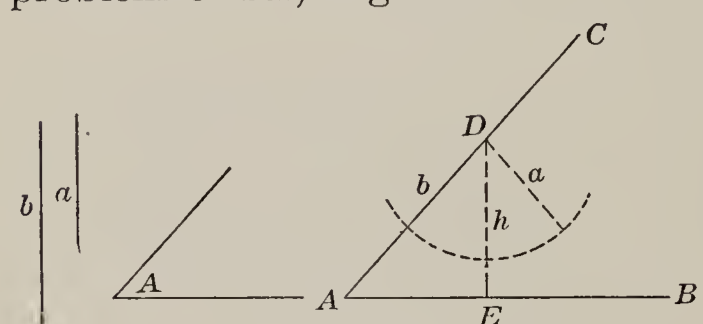


Fig. 85

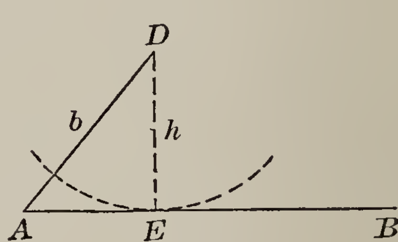


Fig. 86

(2) If $a=h$ the circle will touch AB and there is *one* solution of the problem, i.e., $\triangle ADE$, Fig. 86.

(3) If $a>h$, and $<b$, the circle will intersect AB in *two* points F and F'. There are *two* solutions, i.e., $\triangle ADF$ and $\triangle ADF'$, Fig. 87.

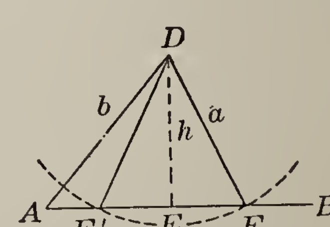


Fig. 87

(4) If a is equal to b the circle will meet AB in A and in another point, F. There is *one* solution, i.e., $\triangle ADF$, Fig. 88.

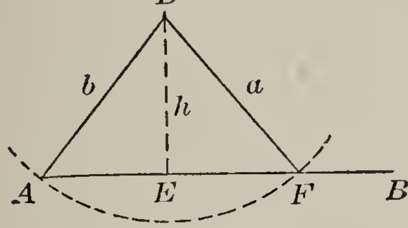


FIG. 88

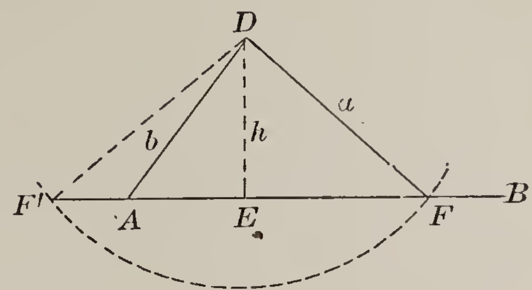


FIG. 89

(5) If $a>b$, the circle will meet AB in two points F and F', but only $\triangle ADF$ satisfies the conditions of the problem, Fig. 89.

Since the length of the perpendicular, h, may be expressed trigonometrically in terms of two of the given parts by means of the equation $h=b \sin A$, the preceding discussion may be summarized briefly as follows:

1. $A>90°$; then $a>b$; *one* solution: an obtuse triangle.
2. $A=90°$; *one* solution: a right triangle
3. $A<90°$; and if $a<b \sin A$; *no* solution

if $a=b \sin A$; *one* solution

if $b>a>b \sin A$; *two* solutions

if $a \geqq b$; *one* solution

Case II is called the **ambiguous case.**

EXERCISES

State, without solving, how many solutions are possible if the given parts are as follows:

1. $A=50°42'$, $a=204$, $b=204$

2. $A=20°10'.3$, $a=57$, $b=42$

3. $A=74°18'13''$, $a=20$, $b=75$

4. $A=32°6'$, $a=802$, $b=785$

5. $A=45°$, $a=108$, $b=152.71$

6. $A=77°17'.6$, $a=210$, $b=196$

Solve the following triangles:

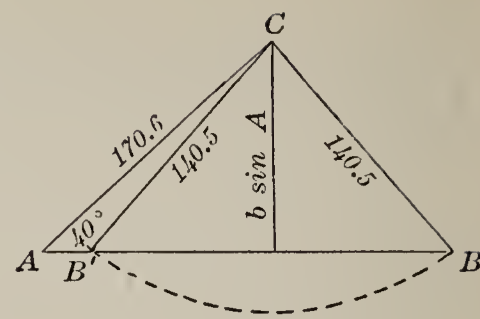


FIG. 90

7. $a=140.5, b=170.6, A=40°$

Discussion:

$$\begin{aligned} \log b &= 2.23198 \\ \log \sin A &= 9.80807-10 \\ \hline \log b \sin A &= 1.03005 \\ \log a &= 2.14768 \end{aligned}$$

$\therefore a>b \sin A$ and there are two solutions, $\triangle ABC$ and $AB'C$, Fig. 90.

Formulas: $\sin B=\dfrac{b \sin A}{a}$, $C=180°-(A+B)$, $c=\dfrac{a \sin C}{\sin A}$

$$b-a=\frac{c \sin \frac{1}{2}(B-A)}{\cos \frac{1}{2} C}$$

Solution:

$$\begin{aligned} \log b &= 2.23198 \\ \log \sin A &= 9.80807-10 \\ \operatorname{colog} a &= 7.85232-10 \\ \hline \log \sin B &= 9.89237-10 \\ B &= 51°18'.4 \\ \therefore B' = \angle AB'C &= 128°41'.6 \\ B+A &= 91°18'.4 \\ C=\angle ACB &= 88°41'.6 \\ B'+A &= 168°41'.6 \\ \therefore C'=\angle ACB' &= 11°18'.4 \end{aligned}$$

$$\begin{aligned} \log a &= 2.14768 \\ \log \sin C &= 9.99989-10 \\ \operatorname{colog} \sin A &= 0.19193 \\ \hline \log c &= 2.33940 \\ c &= 218.49 \\ \log a &= 2.14768 \\ \log \sin C' &= 9.29239-10 \\ \operatorname{colog} \sin A &= 0.19193 \\ \hline \log AB' &= 1.63200 \\ c' = AB' &= 42.855 \end{aligned}$$

Check:

$$\begin{aligned} B-A &= 11°18'.4 \\ \tfrac{1}{2}(B-A) &= 5°39'.2 \\ \tfrac{1}{2}C &= 44°20'.8 \\ b-a &= 30.1 \\ \log(b-a) &= 1.47857 \end{aligned}$$

$$\begin{aligned} \log c &= 2.33940 \\ \log \sin \tfrac{1}{2}(B-A) &= 8.99348-10 \\ \operatorname{colog} \cos \tfrac{1}{2}C &= 0.14562 \\ \hline \log(b-a) &= 1.47850 \end{aligned}$$

$$\begin{aligned} B'-A &= 88°41'.6 \\ \tfrac{1}{2}(B'-A) &= 44°20'.8 \\ \tfrac{1}{2}C' &= 5°39'.2 \end{aligned}$$

$$\begin{aligned} \log c' &= 1.63200 \\ \log \sin \tfrac{1}{2}(B'-A) &= 9.84447-10 \\ \operatorname{colog} \cos \tfrac{1}{2}C' &= 0.00212 \\ \hline \log(b-a) &= 1.47859 \end{aligned}$$

8. $a=491.2, \quad c=385.7, \quad C=46°15'$

9. $a=629, \quad c=462, \quad A=46°10'$

10. $a=723, \quad c=483, \quad A=140°11'$

11. $a=342.6$, $b=745.9$, $A=43°35'.6$

12. $a=345.46$, $b=531.75$, $A=26°47'32''$

13. In the triangle ABC,

$$A=37°21',\ a=93,\ b=85,$$

find the angle B. (Sheffield.)

14. A road OA is $9\frac{5}{8}$ mi. long and makes an angle of $31°16'$ [$=31°.27$] with a straight beach OX. From the point A two straight roads, AB and AB', each 6 mi. long, run to the beach. Find the distance along the beach from O to the nearer of the points B and B'. (Harvard.)

190. Case III. Given two sides and the included angle. The following example illustrates the method:

Given: $B=37°33'40''$, $c=95{,}721$, $a=25{,}463$

Required: A, C, and b.

Formulas: The equation $\tan\frac{1}{2}(C-A)=\dfrac{c-a}{c+a}\cot\frac{1}{2}B$ determines $\frac{1}{2}(C-A)$.

The equation $\frac{1}{2}(C+A)=90°-\frac{1}{2}B$ determines $\frac{1}{2}(C+A)$.

$$C=\tfrac{1}{2}(C+A)+\tfrac{1}{2}(C-A)$$
$$A=\tfrac{1}{2}(C+A)-\tfrac{1}{2}(C-A)$$
$$b=\frac{c\sin B}{\sin C},\quad \frac{c-a}{b}=\frac{\sin\frac{1}{2}(C-A)}{\cos\frac{1}{2}B}$$

Solution:

$$c-a=70{,}258$$
$$c+a=121{,}184$$
$$\tfrac{1}{2}B=18°46'50''$$
$$\tfrac{1}{2}(C+A)=71°13'10''$$

$\log(c-a)=4.84670$
$\text{colog}(c+a)=4.91656-10$
$\log\cot\frac{1}{2}B=0.46847$

$\log\tan\frac{1}{2}(C-A)=0.23173$
$\frac{1}{2}(C-A)=59°36'30''$

$\frac{1}{2}(C+A)=71°13'10''$
$C=130°49'40''$
$A=11°36'40''$

$\log c=4.98100$
$\log\sin B=9.78505-10$
$\text{colog}\sin C=0.12110$

$\log b=4.88715$
$b=77117$

Check: $\log b=4.88715$
$\log\sin\frac{1}{2}(C-A)=9.93580-10$
$\text{colog}\cos\frac{1}{2}B=0.02376$

$\log(c-a)=4.84671$

EXERCISES

Solve the following triangles and check:

1. $a=748$, $b=375$, $C=63°35'30''$
2. $a=486$, $b=347$, $C=51°36'$
3. $a=34.645$ $b=22.531$, $C=43°31'$
4. $a=145.9$, $b=39.90$, $C=92°11'18''$
5. $a=540$ $b=420$, $C=52°6'$
6. $a=469.71$, $b=264.37$, $C=96°57'48''$
7. $a=103.21$, $b=152.37$, $C=141°8'54''$
8. $a=167.38$, $b=152.37$, $C=150°20'6''$

Solve the following problems:

9. The diagonals of a parallelogram are 83.66 and 92.84 and one of the angles of their intersection is 84°.28. Find the sides and angles.

10. Two trains start from the same station at the same time, one going north at 40 mi. per hour, the other going 10° south of east at 30 mi. per hour. How far apart will the trains be at the end of three quarters of an hour? (Harvard.)

11. An aeroplane is observed at the same instant from two stations on a level plane, 5,280 ft. apart. At the first station the horizontal angle between the aeroplane and the other station is 18°37′[=18°.62], and the angle of elevation of the aeroplane is 37°41′[=37°.68]. At the second station the horizontal angle between the first station and the aeroplane is 64°16′[=64°.27]. Find the height of the aeroplane.

By the *horizontal angle* between two points, A and B, each viewed from a point C, is meant the angle between the vertical plane through C and A, and the vertical plane through C and B. (Harvard.)

12. The two diagonals of a parallelogram are 122 and 44, and they form an angle of 47°28′[=47°.47]. Find the lengths of the sides and the angles of the parallelogram. (Harvard.)

13. In a square $ABCD$ a circular arc BD is described, with A as center, and AB as radius. If $AB=5$ ft., find the distance from the point C to one of the points of trisection of the arc BD. (Harvard.)

191. Case IV. Given the three sides. The following example illustrates the method:

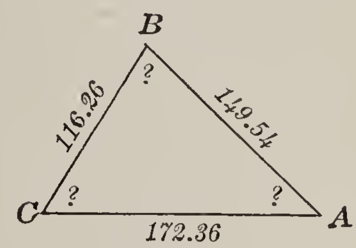


FIG. 91

Given: $a=116.26$, $b=172.36$, $c=149.54$, Fig. 91.

Required: A, B, and C

Formulas: $s=\frac{1}{2}(a+b+c)$, $r=\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$

$$\tan\tfrac{1}{2}A=\frac{r}{s-a},\quad \tan\tfrac{1}{2}B=\frac{r}{s-b},\quad \tan\tfrac{1}{2}C=\frac{r}{s-c}$$

$$A+B+C=180^\circ$$

Solution:

$$\begin{aligned} 2s&=438.16\\ \hline s&=219.08\\ s-a&=102.82\\ s-b&=46.72\\ s-c&=69.54\\ \hline \textit{Check:}\quad 2s&=438.16 \end{aligned}$$

$$\begin{aligned} \log(s-a)&=2.01208\\ \log(s-b)&=1.66950\\ \log(s-c)&=1.84223\\ \operatorname{colog} s&=7.65940-10\\ \hline \log r^2&=3.18321\\ \log r&=11.59160-10 \end{aligned}$$

$$\begin{aligned} \log r&=11.59160-10\\ \log(s-a)&=2.01208\\ \hline \log\tan\tfrac{1}{2}A&=9.57952-10\\ \tfrac{1}{2}A&=20^\circ 47' 42''\\ A&=61^\circ 35' 24'' \end{aligned}$$

$$\begin{aligned} \log r&=11.59160-10\\ \log(s-b)&=1.66950\\ \hline \log\tan\tfrac{1}{2}B&=9.92210-10\\ \tfrac{1}{2}B&=39^\circ 53' 18''\\ B&=79^\circ 56' 36'' \end{aligned}$$

$$\begin{aligned} \log r&=11.59160-10\\ \log(s-c)&=1.84223\\ \hline \log\tan\tfrac{1}{2}C&=9.74937-10\\ \tfrac{1}{2}C&=29^\circ 18' 54''\\ C&=58^\circ 37' 58'' \end{aligned}$$

Check:

$$A+B+C=179^\circ 59' 58''$$

EXERCISES

Solve the following triangles:

1. $a=13,\quad b=14,\quad c=15$
2. $a=77,\quad b=123,\quad c=130$
3. $a=5.953,\quad b=9.639,\quad c=14.424$
4. $a=286,\quad b=321,\quad c=463$
5. $a=12.653,\quad b=17.213,\quad c=23.106$
6. $a=34.278,\quad b=25.691,\quad c=30.175$
7. $a=1{,}017,\quad b=246.3,\quad c=495.98$
8. $a=49.17,\quad b=52.82,\quad c=61.34$

Area of an Oblique Triangle

192. There are various formulas for finding the area of a triangle. Some of them are known to the student from geometry. In computing the area of an oblique triangle, the selection of the formula depends on the given parts.

1. *Given one side and the altitude to that side:*

In this case

$$T=\frac{b\cdot h}{2}$$

where T denotes the area, b a side, and h the altitude to that side.

2. *Given two sides and the included angle:*

Since $T=\frac{1}{2}bh$ and since $h=a\sin C$, it follows that

$$T=\frac{ab\sin C}{2}$$

3. *Given the three sides:*

$$T=\sqrt{s(s-a)(s-b)(s-c)}$$

4. *Given one side and the angles:*

From the law of sines, $a=\dfrac{b \sin A}{\sin B}$.

Substituting in the equation $T=\dfrac{ab \sin C}{2}$,

we have
$$T=\frac{b^2 \sin A \sin C}{2 \sin B}$$

5. *Given the sides and the radius of the inscribed circle:*

$$T=rs$$

6. *Given the sides and the radius of the circumscribed circle:*

We have seen that $\dfrac{c}{\sin C}=2R$, § 180.

$$\therefore \quad \sin C=\frac{c}{2R}.$$

Substituting this in the equation

$$T=\tfrac{1}{2}ab \sin C$$

gives
$$T=\tfrac{1}{2}ab \cdot \frac{c}{2R}$$

$$\therefore \quad T=\frac{abc}{4R}$$

EXERCISES

Find the area of each of the following triangles:

1. $a=40,\quad b=13,\quad c=37$

2. $a=10,\quad b=12,\quad C=60°$

3. $a=122.5,\quad c=122.5,\quad B=110°31'$

4. $A=61°30',\quad B=44°15',\quad c=163$

5. $a=17,\quad b=113,\quad c=120$

Solve the following problems:

6. A regular pentagon is 7 in. on a side. Find the area of the regular five-pointed star obtained by extending the sides of the pentagon. (Harvard.)

7. Prove that the area of a parallelogram is equal to the product of the altitudes divided by the sine of an angle of the parallelogram. (Harvard.)

8. The sides of a parallelogram are, respectively, 28.26 and 30.15. The angle included between them is 68°29. Find the area. (Sheffield.)

9. The side of a rhombus is 2 in. and one angle is 65°. Find the area.

10. In a circle whose radius is 111.3 ft., find the area included between a chord whose length is 129.3 ft., and a diameter parallel to it. (Board.)

11. The diagonals of a parallelogram are 12.5 ft. and 12.8 ft. respectively, and their included angle is 52°16′. Find the sides and the area of the parallelogram.

193. Historical sketch of trigonometry. The *Rhind Papyrus* of somewhere from 2000 to 1700 B.C., sometimes called the *Reckoning Book of Ahmes*, employs the term *seqt* as a technical term having the sense of our word *cosine*.

In his *Risings of the Stars*, Hypsicles of Alexandria (about 180 B.C.) employs calculatory processes with the aid of *chord-functions* of angles. It seems probable that he drew his knowledge of this prototype of our method of reckoning with the *sine-function* from the Babylonians.

Trigonometry is commonly said to have begun as a branch of astronomy with Hipparchus (b. about 160 B.C.). Sometime before 126 B.C. he calculated for his help on astronomical problems a table of chords of circles that was used much as we use a table of sines and for the same purposes.

Hero of Alexandria (first century B.C.), in his work on calculating areas of regular polygons, used a *multiplier* that amounts

to using the *tangent-function*, but as he regarded the method as a part of astronomy and since he was a surveyor, he gave the method little attention. He probably drew his suggestion from ancient Egyptian sources.

Menelaus about 98 A.D. wrote a treatise of six books on chords in circles and made extensive use of calculatory processes based on the *chord-function.*

Ptolemy of Alexandria between 125 and 161 A.D. wrote an epoch-making treatise on astronomy, the *Megiste syntaxis*, in a chapter of which he revived Hipparchus' treatment of the calculation of triangles, including his table of chords. Ptolemy systematized the former treatment, added a few minor improvements, and made considerable use of the tables in calculating, much as we use tables of *natural sines.* This great work, which stood for 1,500 years as the unquestioned authority on astronomy, was translated into the Arabic tongue in the latter half of the ninth century under the Arabic title, the *Almagest.* The *Almagest* was accepted as the authority on trigonometry until 1464 A.D., when Regiomontanus published his *De Triangulis.*

The Hindus on Trigonometry

Aryàbhatta (b. 476 A.D.) is the earliest Hindu scientist whose writings on mathematics have come down to us. His work, the *Aryabhattayam*, furnishes us an example of a highly developed trigonometry employing technical terms for *sine*, *cosine*, and *versed sine* ($=1-\cos$), and containing a table of sines of angles for the interval 3°45′.

Brahmagupta (b. 598) published a book entitled *Siddhanta*, a part of which was devoted to trigonometry, and which, aside from a few improvements in problems over which Aryabhatta had blundered, was no advance upon the older work of Aryabhatta. He included Aryabhatta's table of sines in this work.

Bhaskara (b. 1114 A.D.), another Hindu, calculated and published a more accurate table of sines for angular intervals of 1°.

Arabian Trigonometry

The Arabs were particularly devoted to medicine and astronomy. They were not highly original, but were noted borrowers. They exercised the best of judgment in what they borrowed, drawing from Greek sources in the West and from Hindu sources in the East. They translated the best from both sources into their own tongue, but met with indifferent success in organizing it into a unified body of doctrine. It was the irony of fate that they met with little or no success in advancing either of their pet sciences, nor were they highly successful even in trigonometry, which is closely allied to astronomy.

Alchwarizmi, about 820 A.D., translated an extract from the Hindu *Siddhanta*, and called his translation the *Sindhind*. This *Sindhind* was essentially only the tables of Aryabhatta. It soon gained wide circulation and through it Arabian scientists became acquainted with the Hindu calculatory processes of trigonometry.

Al Battani of Damascus (b. 929 A.D.) was the most significant Arabian trigonometrician. He improved the calculatory procedures of the *Sindhind* and published the results of his work in a book entitled *Stellar Motions*, which was translated into Latin by Plato of Tivoli at the beginning of the twelfth century of our era. This translation became the basis of the work of Regiomontanus.

Abu'l Wafa of Bagdad (940–998) made extensive use of the known trigonometric functions, and also introduced the *tangent* and *cotangent*. He calculated tables of the trigonometric functions including tangents and cotangents for angles advancing by 15′-intervals.

More than a century after Al Battani's book, a Western Arab at Seville in Spain, Dschabir ibn Aflah (between 1140 and 1150) was still using the old Ptolemaic chord-functions in astronomical calculations. He seems to have paid not the least attention to the more effective procedures of either Al Battani or Abu'l Wafa. An anonymous work containing Dschabir's methods became widely disseminated through Europe. It is said that Copernicus made most of the calculations on which his conclusions as to the

plan of the solar system were based, after the ancient Ptolemaic fashion of Dschabir's book, and that the great Copernicus only very gradually and late in the progress of his work adopted Regiomontanus' new methods as they had been developed by Al Battani.

Mediaeval European Trigonometry

Johann Mueller (1436–76), whose scientific *nom de plume* was *Regiomontanus*, while a student of mathematics at the University of Vienna, undertook the translation into Latin and an analysis of the *Almagest*, which had probably been obtained from the Moorish schools in Spain. Regiomontanus, while working on the *Almagest*, became impressed with the importance of getting from Byzantium the original Greek text from which the Arabic translation had been made. This he did, and in his *De Triangulis* of 1464 he gave the results of his work on the trigonometry of the *Megiste syntaxis*, the *Almagest*, the book of Al Battani, together with his own original contributions. The *De Triangulis* was the first European trigonometry as such, and it was the book that did as much as any other one influence to bring about the renaissance of mathematical activity in Europe. Based on all the significant work that had previously been done, *De Triangulis* nevertheless treats the subject differently from any of the sources and in a highly satisfactory form.

Other Europeans before Regiomontanus had worked on one phase or another of trigonometry, and indeed Vieta (1540–1603) had made considerable use of the branch known now as *goniometry*. Rheticus (1514–76) had even suggested the advisability of considering the trigonometric functions *as ratios*. But to Regiomontanus belongs the honor of a first complete scientific treatment in the Western world.

Pitiscus (1561–1613) was the first to use the word *trigonometry* as the title of a book (in 1595).

After Regiomontanus, trigonometry underwent several minor improvements and extensions until under the masterful hand of Euler (1707–83) it assumed its final modern form.

In conclusion, it is one of the curious facts of history that men developed even to a high state of perfection the science of *spherical trigonometry* long before they gave much attention to *plane trigonometry.* Indeed, plane trigonometry was developed to supply a sound scientific basis for a well-wrought-out spherical trigonometry (Tropfke, Band II, S. 189–200).

Summary

194. The following problems review the essential parts of the chapter:

1. What is meant by the logarithms of the trigonometric functions?

2. Explain how to find the value of the logarithm of a function of a given angle.

3. Explain how to find the angle corresponding to a given logarithmic trigonometric function.

4. Discuss the use of logarithms in the solution of right triangles.

5. State the formulas used to solve right triangles.

6. Explain how to solve an isosceles triangle, a regular polygon.

7. State and prove the following laws: law of sines; law of cosines; law of tangents; Mollweide's equations.

8. Give the formulas expressing the following: radius of the circle circumscribed about a triangle; radius of the inscribed circle; tangent of half an angle of a triangle; area of a triangle.

9. Explain the meaning and the use of cologarithms.

10. Discuss the cases occurring in the solution of oblique triangles and state the formulas used in each case.

CHAPTER IX

RELATIONS BETWEEN FUNCTIONS OF SEVERAL ANGLES

Addition and Subtraction Theorems

195. Addition theorems for sine and cosine. The discussion of these theorems is divided into three steps, as follows:

I. Let α and β, Fig. 92, represent two *positive acute* angles whose *sum* is *less than* 90°.

Let $\angle ABC = \alpha + \beta$

To find $\sin(\alpha+\beta)$.

Draw $CD \perp AB$.

Then $\sin(\alpha+\beta) = \dfrac{DC}{CB}$

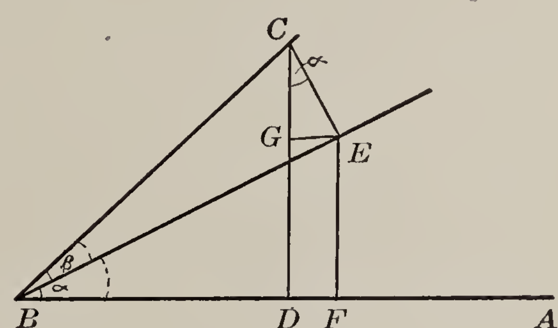


FIG. 92

Draw $CE \perp BE$, $EF \perp AB$ and $EG \perp CD$

Show that

$$\sin(\alpha+\beta) = \frac{DC}{CB} = \frac{DG+GC}{CB} = \frac{FE+GC}{CB} = \frac{FE}{CB} + \frac{GC}{CB}.$$

Show that

$$FE = EB \sin \alpha \text{ and that } \frac{FE}{CB} = \frac{EB}{CB} \sin \alpha = \cos \beta \sin \alpha.$$

Since $\angle GCE = \alpha$, it follows that

$$GC = CE \cos \alpha \text{ and that } \frac{GC}{CB} = \frac{CE}{CB} \cos \alpha = \sin \beta \cos \alpha.$$

By substitution,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{1}$$

$\cos(\alpha+\beta)$ may be found as follows:

$$\cos(\alpha+\beta) = \frac{BD}{CB} = \frac{BF-DF}{CB} = \frac{BF}{CB} - \frac{DF}{CB} = \frac{BF}{CB} - \frac{GE}{CB}$$

Show that

$$BF = EB\cos\alpha \text{ and that } \frac{BF}{CB} = \frac{EB}{CB}\cos\alpha = \cos\beta\cos\alpha$$

Similarly,

$$GE = CE\sin\alpha, \text{ and } \frac{GE}{CB} = \frac{CE}{CB}\sin\alpha = \sin\beta\sin\alpha$$

By substitution,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{2}$$

Equations (1) and (2) are the **addition theorems** for the sine and cosine.

II. Equations (1) and (2) can be proved for $\alpha+\beta$ *greater than* 90°, Fig. 93.

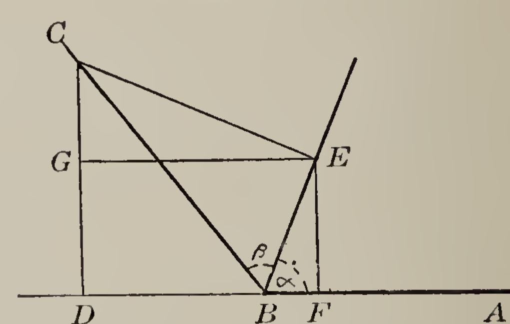


Fig. 93

Equation (1) may be proved exactly as in case I.

To prove equation (2), put

$$\cos(\alpha+\beta) = -\frac{DB}{CB} = -\frac{DF-BF}{CB}, \text{ etc.}$$

It follows that equations (1) and (2) hold for *any two acute angles*.

III. Let one of the angles in equations (1) and (2) be increased by 90°.

Denote the sum by α', i.e., let $\alpha' = 90° + \alpha$.

Then $\sin(\alpha'+\beta)=\sin[90°+(\alpha+\beta)]=\cos(\alpha+\beta)$
$=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$

But $\cos\alpha=\sin(90°+\alpha)=\sin\alpha',$

and $\sin\alpha=-\cos(90°+\alpha)=-\cos\alpha'.$

By substitution, $\sin(\alpha'+\beta)=\sin\alpha'\cos\beta+\cos\alpha'\sin\beta.$

Thus equation (1) holds if one of the angles is increased by 90°.

Similarly,

$$\cos(\alpha'+\beta)=\cos[90°+(\alpha+\beta)]=-\sin(\alpha+\beta)$$
$$=-\sin\alpha\cos\beta-\cos\alpha\sin\beta.$$

By substitution, $\cos(\alpha'+\beta)=\cos\alpha'\cos\beta-\sin\alpha'\sin\beta.$

Thus equations (1) and (2) hold, not only for *any two* acute angles, but they are valid if one or the other angle be increased by 90°.

By the same reasoning it follows that these equations are true if we *repeatedly* increase one or the other angle by 90°. Hence they hold for *any two angles whatever.*

EXERCISES

1. Compute $\sin 75°$; $\cos 75°$

Put $75°=45°+30°$

Then $\sin 75°=\sin(45°+30°)=\sin 45°\cos 30°+\cos 45°\sin 30°$

$$=\tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}.$$

Compute the sine and cosine of each of the following angles:

2. 15°

Put $15°=60°-45°$.

3. 90°

Put $90°=60°+30°$.

4. 105°

Put $105°=60°+45°$.

5. 150°

6. 210°

7. 195°

8. 120°

9. 240°

Verify the following:

10. $\sin(45°+x)=\dfrac{(\cos x+\sin x)\sqrt{2}}{2}$

11. $\cos(60°+a)=\dfrac{\cos a-\sqrt{3}\sin a}{2}$

196. Subtraction theorems for sine and cosine. These theorems are proved as follows:

Let $\alpha=(\alpha-\beta)+\beta$

Show that,

$$\sin\alpha=\sin[(\alpha-\beta)+\beta]$$
$$=\sin(\alpha-\beta)\cos\beta+\cos(\alpha-\beta)\sin\beta,$$
$$\cos\alpha=\cos[(\alpha-\beta)+\beta]$$
$$=\cos(\alpha-\beta)\cos\beta-\sin(\alpha-\beta)\sin\beta.$$

Denoting $\sin(\alpha-\beta)$ by x and $\cos(\alpha-\beta)$ by y, these equations take the form

$$\sin\alpha=x\cos\beta+y\sin\beta$$

and $$\cos\alpha=-x\sin\beta+y\cos\beta$$

Solving by determinants,

$$x=\frac{\begin{vmatrix}\sin\alpha & \sin\beta\\ \cos\alpha & \cos\beta\end{vmatrix}}{\begin{vmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{vmatrix}},\quad y=\frac{\begin{vmatrix}\cos\beta & \sin\alpha\\ -\sin\beta & \cos\alpha\end{vmatrix}}{\begin{vmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{vmatrix}}$$

$$\therefore\quad x=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta,}{\cos^2\beta+\sin^2\beta},$$

$$y=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos^2\beta+\sin^2\beta}$$

$$\therefore\quad x=\sin\alpha\cos\beta-\cos\alpha\sin\beta,$$

$$y=\cos\alpha\cos\beta+\sin\alpha\sin\beta,$$

or $$\boldsymbol{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

and $$\boldsymbol{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.}$$

EXERCISES

1. Compute sin 15°; cos 15°

$$\sin 15° = \sin (45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Compute the sine and cosine of the following angles:

2. 0°

Put $0° = 30° - 30°$

3. 15°

Put $15° = 60° - 45°$

Apply the subtraction theorems in the following:

4. $\sin (90° - x)$

5. $\cos (45° - x)$

6. $\cos (30° - y)$

7. $\sin (360° - y)$

Verify the following:

8. $\sin (x+y) \sin (x-y) = \sin^2 x - \sin^2 y$

9. $\cos (x+y) \cos (x-y) = \cos^2 x - \cos^2 y$

10. $\sin (x+y+z) = \sin x \cos y \cos z + \cos x \sin y \cos z$
$\qquad + \cos x \cos y \sin z - \sin x \sin y \sin z$

197. Sums and differences of sines and cosines. By means of the formulas developed below, the sums or differences of sines and cosines can be transformed into products.

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \qquad (1)$$
$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \qquad (2)$$

Adding (1) and (2),

$$\sin (\alpha + \beta) + \sin (\alpha - \beta) = 2 \sin \alpha \cos \beta \qquad (3)$$

Subtracting (2) from (1),

$$\sin (\alpha + \beta) - \sin (\alpha - \beta) = 2 \cos \alpha \sin \beta \qquad (4)$$

Similarly,

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta \tag{5}$$

and

$$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta \tag{6}$$

Let $\alpha+\beta=A$ and $\alpha-\beta=B$

Then $\alpha=\frac{1}{2}(A+B)$ and $\beta=\frac{1}{2}(A-B)$ Why?

Substituting these results into equations (3) to (6),

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

$$\cos A-\cos B=-2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

EXERCISES

Verify the following:

1. $\sin 35°+\sin 15°=2\sin 25°\cos 10°$

2. $\sin 75°+\sin 15°=\frac{1}{2}\sqrt{6}$

3. $\sin 3\theta+\sin\theta=2\sin 2\theta\cos\theta$

4. $\sin 5x+\sin 3x=2\sin 4x\cos x$

5. $\sin\frac{3x}{2}-\sin\frac{x}{2}=2\cos x\sin\frac{x}{2}$

6. $\sin(45°+x)+\sin(45°-x)=\sqrt{2}\cos x$

7. $\dfrac{\sin 2\theta+\sin\theta}{\cos 2\theta-\cos\theta}=\cot\dfrac{\theta}{2}$

8. $\dfrac{\sin 6x+\sin 4x}{\cos 6x+\cos 4x}=\tan 5x$

198. Addition and subtraction theorems for tangent and cotangent.

Show that

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\therefore\quad \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Similarly,
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

EXERCISES

Verify the following:

1. $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$

2. $\cot(\alpha-\beta)=\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

3. $\tan(45+A)=\dfrac{1+\tan A}{1-\tan A}$

4. $\tan(45-A)=\dfrac{1-\tan A}{1+\tan A}$

199. Functions of double an angle.

In the equation
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,$$
put
$$\beta=\alpha.$$
Then
$$\sin(2\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha,$$
or
$$\sin 2\alpha=2\sin\alpha\cos\alpha.$$

Similarly, from

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta,$$

it follows that

$$\boldsymbol{\cos 2\alpha=\cos^2\alpha-\sin^2\alpha.}$$

Since $\cos^2\alpha=1-\sin^2\alpha$, we have

$$\boldsymbol{\cos 2\alpha=1-2\sin^2\alpha.}$$

By means of the equation

$$\sin^2\alpha=1-\cos^2\alpha$$

we obtain

$$\cos 2\alpha=\cos^2\alpha-(1-\cos^2\alpha)$$
$$=\cos^2\alpha-1+\cos^2\alpha$$

$$\therefore\quad \boldsymbol{\cos 2\alpha=2\cos^2\alpha-1.}$$

From the equation

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

we have

$$\boldsymbol{\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}.}$$

EXERCISES

1. Show that $\cot 2\alpha=\dfrac{\cot^2\alpha-1}{2\cot\alpha}$

2. Show that $\tan 3\alpha=\dfrac{3\tan\alpha-\tan\alpha}{1-3\tan^2\alpha}$

Put $3\alpha=(2\alpha+\alpha)$.

200. Functions of half an angle.

In the equation

$$\cos 2a = 1 - 2\sin^2 a,$$

let $\quad 2a = x$

then $\quad a = \frac{x}{2}.$

This gives the equation

$$\cos x = 1 - 2\sin^2\frac{x}{2}.$$

Solving for $\sin\frac{x}{2}$,

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}.$$

Similarly, from the equation

$$\cos 2a = 2\cos^2 a - 1$$

we derive

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos a}{2}}.$$

By dividing $\sin\frac{x}{2}$ by $\cos\frac{x}{2}$, we have

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}.$$

EXERCISES

Prove that the following statements are identities:

1. $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

$\sin 2x = 2\sin x\cos x$

$$\frac{2\tan x}{1+\tan^2 x} = \frac{2\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin^2 x}{\cos^2 x}} = \frac{2\sin x\cos x}{\cos^2 x+\sin^2 x} = \frac{2\sin x\cos x}{1}$$

2. $\left(\sin \frac{A}{2}+\cos \frac{A}{2}\right)^2=1+\sin A$

3. $\frac{\sin 2a}{1+\cos 2a}=\tan a$

4. $\tan A+\cot A=2 \csc 2A$

5. $\frac{\tan x+\tan y}{\cot x+\cot y}=\tan x \tan y$

6. $\cot x=\frac{\sin 2x}{1-\cos 2x}$

7. $\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}=\cos x$

8. $1+\tan A \tan \frac{A}{2}=\sec A$

MISCELLANEOUS EXERCISES

201. Verify the following statements:

1. $\tan 2x+\sec 2x=\frac{\cos x+\sin x}{\cos x-\sin x}$

2. $2 \sin x+\sin 2x=\frac{2 \sin^3 x}{1-\cos x}$

3. $(\sec a+\tan a)^2=\frac{1+\sin a}{1-\sin a}$

4. $\sin (\frac{1}{3}\pi+a)-\sin (\frac{1}{3}\pi-a)=\sin a$

5. $\cos 3x=4 \cos^3 x-3 \cos x$

6. $1+\tan a \tan \frac{a}{2}=\sec a$

7. $\tan x=\tan \frac{x}{2}+\tan \frac{x}{2} \sec x$

8. $\frac{\cos (a-45^\circ)}{\cos (a+45^\circ)}=\frac{\cos 2a}{1-\sin 2a}$

Solve the following problems:

9. Given $\tan x = -\frac{4}{3}$ and x in the second quadrant. Find $\sin 2x$. (Sheffield.)

10. $\tan x = \frac{7}{24}$ and x is in the third quadrant. $\sec y = -\frac{13}{5}$ and y is in the second quadrant. Find $\cos (x-2y)$; $\cot 2x$; $\sin \frac{1}{2}y$. (Board.)

11. If $\sin x = \dfrac{2mn}{m^2+n^2}$, find $\tan \dfrac{x}{2}$. (Harvard.)

12. If $\tan \dfrac{x}{2} = y$, find the values of $\sin x$ and $\cos s$ in terms of y. (Harvard.)

13. Prove for a right triangle that the cosine of the difference between the acute angles is equal to twice the product of the two legs, divided by the square of the hypotenuse. (Harvard.)

14. If $x = \tan^{-1} \dfrac{2a}{1-a^2}$, find the value of $\sin \dfrac{x}{2}$ in terms of a (consider only values of x between $0°$ and $90°$, and values of a between 0 and 1). (Harvard.)

15. Solve the equation

$$\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3. \quad \text{(Sheffield.)}$$

Let $\alpha = \tan^{-1} x$, $\beta = \tan^{-1} 2x$.
Then $\tan (\alpha + \beta) = 3$, etc.

16. Prove $\text{arc} \tan \frac{1}{2} + \text{arc} \tan \frac{1}{3} = 45°$.

Trigonometric Equations

202. Solve the following equations for values of x between $0°$ and $360°$:

1. $\cos 2x + 3 \sin x = 2$

Show that $1 - 2 \sin^2 x + 3 \sin x = 2$.
Solve for $\sin x$. Then find x.

2. $\cos 2x + \cos x = 0$

3. $\cos x \cos 2x + 2 \cos^3 x = 0$

Solve by factoring.

4. $\tan\left(\frac{\pi}{4}+x\right)+\tan\left(\frac{\pi}{4}-x\right)=4$

Expand each term.

5. $\sin x+\sin 2x=1$

6. $\cos 2x+\sin x=4\sin^2 x$

‡7. Solve the equation

$$3\sec^2 x-7\tan^2 x=\tan x.$$

Obtain all solutions for x between 0° and 180° and give the answers to the nearest degree. (Yale.)

‡8. Solve the equation

$$\sin 2x+\tfrac{1}{2}=\sin x+\cos x.$$

Obtain all solutions for x between 0° and 180° and give the answers in degrees. (Yale.)

‡9. Find all the values of x between 0° and 360° which satisfy the equation

$$4\cos 2x+3\cos x=1. \quad \text{(Harvard.)}$$

‡10. Find all the values of x between 0° and 360° which satisfy the equation

$$6\cos 2x+6\sin^2 x=5+\sin x,$$

and verify your results. (Harvard.)

‡11. Find all the values of x between 0° and 360° which satisfy the equation

$$3\cos 2x+\sin x\,(3\sin x+5)=5. \quad \text{(Harvard.)}$$

‡12. Find all the values of x between 0° and 360° for which

$$2\sin 2x=\cos x. \quad \text{(Harvard.)}$$

‡13. The sum of the tangents of the acute angles of a right triangle is equal to 4. Find the values of these angles. (Harvard.)

‡14. Solve the equation

$$\cos 5\theta+\cos 3\theta=\sqrt{2}\cos 4\theta. \quad \text{(Princeton.)}$$

Summary

203. The following formulas have been proved:

1. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
2. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
3. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
4. $\cos(\alpha-\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
5. $\sin\alpha+\sin\beta=2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
6. $\sin\alpha-\sin\beta=2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$
7. $\cos\alpha+\cos\beta=2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
8. $\cos\alpha-\cos\beta=-2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$
9. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
10. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
11. $\sin 2\alpha=2\sin\alpha\cos\alpha$
12. $\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$
    $=2\cos^2\alpha-1$
    $=1-2\sin^2\alpha$
13. $\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}$
14. $\sin\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1-\cos\alpha}{2}}$
15. $\cos\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1+\cos\alpha}{2}}$
16. $\tan\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

204. The chapter has shown how to solve trigonometric equations. Explain your method of solving such equations.

205. Explain how to prove trigonometric identities.

CHAPTER X

BINOMIAL THEOREM. ARITHMETICAL AND GEOMETRICAL PROGRESSIONS

Binomial Theorem

206. The *binomial theorem* enables us to state by inspection the expansion of a power of a binomial. By actually multiplying, the following identities are obtained:

$$(a+b)^2=a^2+2ab+b^2$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5, \text{ etc.}$$

A study of these identities brings out the following facts:

1. *The number of terms in the right member is one greater than the exponent of the binomial in the left member.*

2. *The exponent of a in the first term of the expansion is the same as the exponent of the binomial and decreases by one in each succeeding term, being 0 in the last term.*

3. *The exponent of b increases by 1 from term to term, being 0 in the first term and the same as the exponent of the binomial in the last term.*

Omitting the coefficients, these three facts give the following expansion of *any binomial*, as $(a+b)$, raised to *any positive integral power*, n:

$$(a+b)^n=a^n+()a^{n-1}b+()a^{n-2}b^2+()a^{n-3}b^3+\dots.$$

The coefficients may be determined by the following simple device:

Arrange the coefficients of $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, etc., as in the form given in Fig. 94. Notice that each coefficient in this arrangement is equal to the sum of the coefficients which are nearest to the right and left of it in the line above.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

FIG. 94

Fig. 94 is known as **Pascal's triangle.***

Moreover, after the second term *any* coefficient may also be determined by means of the coefficient of the term *just preceding*, according to the following rule:

Multiply the coefficient of the preceding term by the exponent of a in that term and divide the product by the number of that term.

Thus in $(a+b)^5$ the coefficient of the fifth term is $\frac{10\cdot 2}{4}=5$; the coefficient of the third term is $\frac{5\cdot 4}{2}=10$; etc.

207. The binomial theorem. According to the rules given in § 206 the expansion of $(a+b)^n$ takes the following form:

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\ldots .$$

This is known as the **binomial theorem** for *positive integral* powers. The theorem is *assumed* without proof. The proof is usually given in a course in advanced algebra.

* See *First-Year Mathematics*, pp. 200–201.

208. The factorial notation. The product $1\cdot 2$, $1\cdot 2\cdot 3$, $1\cdot 2\cdot 3\cdot 4$, ... , $1\cdot 2\cdot 3\cdot 4 \dots r$, are called *factorial* 2, *factorial* 3, *factorial* 4, ... , *factorial* r. They are usually denoted briefly by the symbols: $2!$, $3!$, $4!$, ... $r!$ or by $\underline{|2}$, $\underline{|3}$, $\underline{|4}$, ... , $\underline{|r}$.

$$\text{Thus, } (a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2$$
$$+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+\dots.$$

EXERCISES

Expand the following powers to four terms:

1. $(2x-3y)^7$

In this example $a=2x$, $b=(-3y)$, $n=7$.

$$\text{Hence } (2x-3y)^7=(2x)^7+7(2x)^6(-3y)+\frac{7\cdot\overset{3}{\cancel{6}}}{\cancel{2}}(2x)^5(-3y)^2+$$
$$\frac{\overset{7}{\cancel{21}}\cdot 5}{\cancel{3}}(2x)^4(-3y)^3+\dots$$

$$=2^7x^7-7\cdot 2^6\cdot 3x^6y+21\cdot 2^5\cdot 3^2x^5y^2-35\cdot 2^4 3^3x^4y^3+\dots.$$
$$=128x^7-1344x^6y+6048x^5y^2-15120x^4y^3+\dots.$$

2. $\left(\frac{2}{y}+\frac{3y^2}{4}\right)^4$

Here $a=\frac{2}{y}$, $b=\frac{3y^2}{4}$, $n=4$

$$\therefore\quad \left(\frac{2}{y}+\frac{3y^2}{4}\right)^4=\left(\frac{2}{y}\right)^4+4\left(\frac{2}{y}\right)^3\left(\frac{3y^2}{4}\right)$$
$$+\frac{\overset{2}{\cancel{4}}\cdot 3}{1\cdot\cancel{2}}\left(\frac{2}{y}\right)^2\left(\frac{3y^2}{4}\right)^2+\frac{\overset{2}{\cancel{6}}\cdot 2}{\cancel{3}}\left(\frac{2}{y}\right)\left(\frac{3y^2}{4}\right)^3+\frac{\cancel{4}\cdot 1}{\cancel{4}}\left(\frac{2}{y}\right)^0\left(\frac{3y^2}{4}\right)^4$$
$$=\frac{2^4}{y^4}+\frac{4\cdot 2^3\cdot 3y^2}{4y}+\frac{6\cdot 2^2\cdot 3^2y^4}{4^2y^2}+\frac{4\cdot 2\cdot 3^3y^6}{4^3y}+\frac{3^4y^8}{4^4}$$
$$=\frac{16}{y^4}+24y+\frac{27y^2}{2}+\dots.$$

3. $\left(\frac{2a}{b^2}-b\sqrt{a}\right)^4$

Let $a=\frac{2a}{b^2}$, $b=-b\sqrt{a}=-ba^{\frac{1}{2}}$

Then $\left(\frac{2a}{b^2}-b\sqrt{a}\right)^4=\left(\frac{2a}{b^2}\right)^4+4\left(\frac{2a}{b^2}\right)^3(-ba^{\frac{1}{2}})$

$$+\frac{\overset{2}{\cancel{4}}\cdot 3}{\cancel{2}}\left(\frac{2a}{b^2}\right)^2(-ba^{\frac{1}{2}})^2+\ldots\ldots$$

$$=\frac{2^4a^4}{b^8}-\frac{4\cdot 2^3a^3a^{\frac{1}{2}}b}{b^6}+\frac{6\cdot 2^2a^2ab^2}{b^4}\ldots\ldots$$

$$=\frac{16a^4}{b^8}-\frac{32a^3\sqrt{a}}{b^5}+\frac{24a^3}{b^2}\ldots\ldots\ldots$$

4. $(x-y)^8$
5. $(2a+4c)^4$
6. $(3a^2-b^2)^7$
7. $(3a^2b^4-\frac{1}{2})^5$
8. $\left(\frac{3}{4}x-\frac{4}{3}y\right)^6$
9. $\left(y^2-\frac{b}{2y}\right)^6$
10. $(\sqrt{x}+\sqrt{y})^5$
11. $(2+\sqrt{3})^4$
12. $\left(\frac{\sqrt{x}}{y}-\frac{\sqrt{y}}{x}\right)^8$
13. $\left(\frac{a\sqrt[3]{a}}{\sqrt[4]{b^3}}-\frac{\sqrt[3]{b}}{a}\right)^5$
14. $(x^{\frac{1}{2}}+y^{\frac{2}{3}})^4$
15. $(a^{-3}-b^{-3})^4$
16. $(3ab^{-3}-a^{-3}b)^6$
17. $\left(a^2-\frac{1}{2\sqrt{a}}\right)^{10}$
18. $\left(\frac{3}{2k}-\frac{2k}{3}\right)^4$
19. $\left(y^{\frac{1}{2}}-\frac{2}{\sqrt[3]{y}}\right)^5$
20. $(4a^{\frac{3}{2}}-a^{\frac{1}{2}}b^{\frac{1}{3}})^6$

209. The *r*th term of $(a+b)^n$. The *number* of the term and the *numerator* of the coefficient of the same term may be arranged in the following table:

Number of Term	Numerator
3	$n(n-1)$
4	$n(n-1)(n-2)$
5	$n(n-1)(n-2)(n-3)$
6	$n(n-1)(n-2)(n-3)(n-4)$

From a study of this table show that the numerator of the coefficient of the rth term is

$$n(n-1)(n-2)(n-3).\ .\ .\ .\ (n-r+2).$$

Similarly, find that the *denominator* of the coefficient of the rth term is $1 \cdot 2 \cdot 3 \cdot 4 \cdot\ .\ .\ .\ .\ (r-1)$.

Show that the *exponent of* a in the rth term is $n-r+1$.

Show that the *exponent of* b in the rth term is $r-1$.

Thus the rth term in the expansion of $(a+b)^n$ is given by the formula

$$t_r = \frac{n(n-1)(n-2)\ .\ .\ .\ .\ (n-r+2)}{1 \cdot 2 \cdot 3 \cdot\ \ .\ .\ .\ .\ (r-1)} a^{n-r+1} b^{r-1}.$$

EXERCISES

In the expansions of the following binomials find the term called for in each case:

1. In $\left(2x+\frac{1}{2x}\right)^6$ find the fourth term.

Here $a=2x$, $b=\frac{1}{2x}$, $n=6$, $r=4$

$$\therefore t_4 = \frac{\cancel{6} \cdot 5 \cdot 4}{1 \cdot \cancel{2} \cdot \cancel{3}}(2x)^{6-4+1}\left(\frac{1}{2x}\right)^{4-1}$$

$$= \frac{20 \cdot 2^3 \cdot x^3}{2^3 x^3} = 20$$

2. In $\left(\frac{5}{a}+\frac{a}{5}\right)^8$ find the fifth term.

3. In $(\sqrt{x}+\sqrt[3]{y})^7$ find the sixth term.

4. In $(\frac{1}{2}x-\frac{1}{3}y)^{10}$ find the fourth term.

5. Write the last three terms of the expansion of $(4a^{\frac{3}{2}}-a^{\frac{1}{2}}x^{\frac{1}{3}})^8$. (Yale.)

MISCELLANEOUS EXERCISES

‡**210.** The following problems are taken from college-entrance examination papers:

1. Expand and simplify $\left(\frac{2x}{y^3}-\frac{y^4}{x\sqrt[5]{-6}}\right)^5$. (Smith.)

2. Write the last three terms of the expansion of $(4a^{\frac{3}{2}}-a^{\frac{1}{2}}x^{\frac{1}{3}})^8$. (Yale.)

3. Prove that $(a+b)^7-a^7-b^7=7ab(a+b)(a^2+ab+b^2)^2$. (Harvard.)

4. Find the fifth term of $\left(x^4+\frac{1}{5x^2}\right)^{10}$ and reduce to the simplest form. (Dartmouth.)

5. In the expansion of $\left(2x+\frac{1}{3x}\right)^6$ the ratio of the fourth term to the fifth is 2:1. Find x. (Princeton.)

6. Write the sixth term of $\left(\frac{x}{2\sqrt[3]{y^2}}-\frac{\sqrt{y}}{x}\right)^9$. (Pennsylvania.)

7. Find and simplify the twenty-third term in the expansion of $\left(\frac{2x^2}{3}-\frac{3}{4}\right)^{28}$. (Cornell.)

8. If the middle term of $\left(3x-\frac{1}{2\sqrt{x}}\right)^4$ is equal to the fourth term of $\left(2\sqrt{x}+\frac{1}{2\sqrt{x}}\right)^7$, find the value of x. (M.I.T.)

9. Write the first term of $(x^{\frac{1}{2}}-x^{-\frac{2}{3}})^8$ which in its simplest form has a negative exponent. (Board.)

10. Find the coefficient of x^4 in $\left(2x^2-\frac{1}{4x}\right)^8$.

11. Show that the coefficient of the middle term of $(1+x)^{16}$ is equal to the sum of the coefficients of the eighth and ninth terms of $(1+x)^{15}$. (Princeton.)

12. In the expansion of $(2x-3x^{-1})^8$ find that term which does not contain x. (Princeton.)

13. Write the sixth term in the expansion of

$$\left(\sqrt{\frac{64a^4b^6}{81m^2n^8}}+\sqrt[3]{-\frac{a^{-3}}{m^{-9}}}\right)^{10}. \quad \text{(Yale.)}$$

14. Find the third and fifth terms in the expansion of $(1-\sqrt{x})^6$. (Sheffield.)

15. In the expansion of $\left(\frac{2x^2}{3}-\frac{3}{2x}\right)^8$ find the coefficient of x^4.

16. In the expansion of $\left(a+\frac{1}{a}\right)^{14}$ write the term which does not contain x.

17. Find the middle term in the expansion of $\left(\frac{a}{b}+\frac{b}{a}\right)^8$.

Arithmetical Progression

211. Arithmetical progression. A succession of numbers formed according to a definite law is called a *series*. Thus the expression $2+4+6+8\ldots.$ is a series, the law of formation being that any term in the series is obtained by adding the fixed number, 2, to the preceding term.

An *arithmetical progression* is a succession of numbers in which each term after the first may be found by adding a constant number to the preceding term.

EXERCISES

1. Show that the following are examples of arithmetical progressions: 3, 5, 7, 9. . . . ; 84, 74, 64. . . . ; the logarithms, to the base 3, of the numbers 3, 9, 27, 81. . . .

2. Show that the equation $a-b=b-c$ expresses the fact that three numbers a, b, and c are in arithmetical progression.

212. Elements. In general, the form of an arithmetical progression is

$$a,\ (a+d),\ (a+2d),\ (a+3d),\ldots.$$

The *first term* is denoted by a,
the *constant difference* by d,
the *number of terms* considered by n,
the *nth term* by l,
and the *sum of n terms* by s.

The numbers a, d, n, l, and s are the **elements** of the arithmetical progression.

213. Relations between the elements. The table, Fig. 95, shows one of the relations between the elements of an arithmetical progression.

Number of Term	Term
Second.........	$a+d$
Third..........	$a+2d$
Fourth........	$a+3d$
Fifth..........	$a+4d$
Tenth.........	$a+9d$
Fifteenth.......	$a+14d$
nth............	$a+(n-1)\,d$

$$\therefore \quad l=a+(n-1)d$$

FIG. 95

Another relation may be obtained as follows:

$$s=a+(a+d)+(a+2d)+\;.\;.\;.\;.\;a+(n-1)d$$

Similarly,

$$s=l+(l-d)+(l-2d)+\;.\;.\;.\;.\;l-(n-1)d$$

Adding,

$$2s=(a+l)+(a+l)+(a+l)+\;.\;.\;.\;.\;+(a+l)$$

Combining terms,

$$2s=n(a+l)$$

$$\therefore \quad s=\frac{n}{2}(a+l)$$

The two relations just established enable us to find the values of two of the elements if the other three are known.

EXERCISES

Solve the following problems:

1. Find the tenth term of the series $7+10+13\ldots.$

Let $a=7$, $d=3$, $n=10$.

Substitute these values in the equation $l=a+(n-1)d$, and find the required value of l.

2. Find the seventh term and the sum of seven terms of the series $2+4+6\ldots.$

3. If $a=7$, $d=-2$, and $n=8$, find s and l.

4. If $l=30$, $n=9$, and $s=162$, find a and d.

5. The fourth term of an arithmetical progression is 11 and the fourteenth term is -39. Find the common difference.

6. The sum of the second and twentieth terms of an A.P. is 10, and their product is $23\frac{47}{64}$. What is the sum of 16 terms? (Pennsylvania.)

7. Given $l=23$, $d=2$, $s=143$. Find a and n.

By substituting for l, d, and s the given values,

$$23=a+(n-1)2$$
$$143=\frac{n}{2}(a+23)$$

or

$$a+2n=25$$
$$an+23n=286$$

Solving the first equation for a and substituting into the second,

$$(25-2n)n+23n=286$$

or

$$2n^2-48n+286=0$$
$$\therefore n=11,\ 33$$

Evidently there are two progressions: $3, 5, 7, 9, \ldots.$ and $-41, -39, -37\ldots.$

8. Given $a=5$, $d=3$, and $s=185$. Find l and n.

9. Given $l=33$, $s=152$, and $d=4$. Find a and n.

10. Find the sum of all even integers from 1 to 100.

11. Find the sum of all integers divisible by 3, between 0 and 320.

12. Find the sum of all positive integers of three digits which are multiples of 9.

13. Find the nth term and the sum of n terms in the progression 1, 3, 5, 7....

14. Find the first term if the common difference is 5 and the twenty-seventh term is 139.

15. How many numbers divisible by 6 are between 0 and 200?

16. A man agreed to dig a well at the following rate: for the first yard he was to receive $8 and for every succeeding yard $2 more. If the well was 27 yd. deep, how much did he receive?

17. A man buys a house and lot for the sum of $6,200. He agrees to pay $600 at the end of the first year, $650 at the end of the second year, etc. How many years will it take him to pay for the property?

18. The sum of the first eight terms of an arithmetical progression is 64 and the sum of the first 18 terms is 324. Find the series. (Princeton.)

19. A man takes a position with the understanding that he will receive $800 the first year with an increase of $50 each succeeding year for the next 20 years. What is his salary during the fifteenth year?

20. In a potato race 45 potatoes are placed in a straight line with a basket and 3 ft., 6 ft., 9 ft., 12 ft., etc., from it. What is the total distance a boy must run to carry the potatoes to the basket one at a time?

21. A ball rolling down an inclined plane goes 8 ft. the first second. In each second thereafter it passes over 16 ft. more

than in the preceding second. How far will it roll in 12 seconds?

22. A body starting from rest is observed to fall 16.08 ft. in the first second, 48.24 ft. in the second, 80.40 in the third, etc. How far does it fall in 12 seconds? in 15 seconds? in t seconds?

23. The sum of three terms of an arithmetical progression is 33. The square of the last term exceeds the sum of the squares of the first two by 11. What are the numbers?

24. A bullet is fired directly upward with a speed of 1,800 ft. a second. As it goes up its speed decreases and as it comes down its speed increases by 32 ft. a second. How high will it rise? In what time will it reach the ground?

214. Arithmetical means. The terms of an arithmetical progression between any two other terms are called **arithmetical means.**

For example,

5 is an arithmetical mean between 2 and 8;

9, 5, 1, and -3 are four arithmetical means between 13 and -7.

Between two given numbers, one or more arithmetical means may be inserted.

Thus, if three arithmetical means are to be inserted between 5 and 69, we have the series $5+\ .\ .\ .\ +69$, the three dots indicating the three means to be inserted.

Hence $a=5$, $l=69$, $n=5$, and d is to be found.

From the formula $l=a+(n-1)d$, we have

$$69=5+4d,$$
$$\therefore\ d=16.$$

Hence the three arithmetical means between 5 and 69 are 21, 37, 53.

EXERCISES

1. Insert 8 arithmetical means between -5 and -3.

2. Insert 4 arithmetical means between 9 and 11.

3. Find the arithmetical mean between 15 and 7.

4. How many arithmetical means must be inserted between 249 and 15 to make the sum of the series equal to 1,995?

5. Insert 6 arithmetical means between 8 and 3.

6. Arithmetical means are inserted between 1 and 21 so that the sum of these means is 132. Find the first two of them. (Board.)

Geometrical Progression

215. Geometrical progression. A succession of numbers in which any term after the first is obtained by *multiplying* the preceding term by a fixed number is a *geometrical progression.*

The fixed number is called the **common ratio.**

EXERCISES

1. Show that the following are examples of geometrical progressions and find the common ratio:

1. 3, 6, 12, 24....
2. 27, -9, $+3$, -1....

2. Show that three numbers, a, b, and c, are in geometrical progression if the ratio $\frac{a}{b}$ is equal to the ratio $\frac{b}{c}$, i.e., if $\frac{a}{b}=\frac{b}{c}$.

The general form of geometric progression is

$a, ar, ar^2, ar^3 \ldots,$

where a is the first term and r the common ratio.

216. Elements of a geometrical progression. The first term, a, the nth term, l, the number of terms, n, the common ratio, r, and the sum of n terms, s, are the **elements** of the geometrical progression.

217. Relations between the elements. The table, Fig. 96, shows how one of the relations between the elements may be found.

Number of the Term	Term
Second........	ar
Third..........	ar^2
Fourth........	ar^3
Fifth..........	ar^4
Twelfth........	ar^{11}
nth............	ar^{n-1}

$$\therefore \quad l = ar^{n-1}$$

Fig. 96

A formula for the sum of n terms is worked out as follows:

Let $s = a + ar + ar^2 + ar^3 + \ldots\ldots ar^{n-2} + ar^{n-1}$.

Multiplying by r,

$$rs = ar + ar^2 + ar^3 + ar^4 + \ldots\ldots ar^{n-1} + ar^n.$$

Subtracting the second equation from the first,

$s - rs = a - ar^n$,

or $(1 - r)s = a(1 - r^n)$,

$$\therefore \quad s = \frac{a(1-r^n)}{1-r}.$$

EXERCISES

1. Find the twelfth term of the progression $1, \frac{1}{2}, \frac{1}{4} \ldots$. What is the sum of the first 8 terms?

2. Given $a = 16$, $r = -\frac{1}{2}$, $l = -\frac{1}{8}$. Find n and s.

3. Find the sum of 10 terms of the progression $27, -9, 3, -1 \ldots$

4. Given $a = 5$, $n = 10$, $s = 50$. Find l and r.

5. Find the eighth term of the series $\frac{64}{12}-\frac{16}{3}+4-3+\ldots$

6. The fourth term of a geometrical progression is 32, the eighth term is 512. Find the tenth term.

7. The sum of the first and third terms of a geometrical progression is 40 and the second term is 16. Find each term.

8. Three numbers are in geometrical progression. The second is 32 greater than the first and the third 96 greater than the second. Find the numbers.

9. A chain letter is sent by a person to two friends with the request that each send a copy of the letter to two friends, with a request that they in turn send a copy to each of two friends, etc. After 12 sets of letters have been sent, how many copies of the original letter have been made?

10. An elastic ball bounces to three-fourths of the height from which it falls. If it is thrown up from the ground to a height of 15 ft., find the total distance traveled before it comes to rest. (Board.)

11. The sum of the first 8 terms of a geometrical progression is seventeen times the sum of the first 4 terms. Find the value of the common ratio. (Board.)

12. The sum of the first 10 terms of a geometrical progression is equal to 244 times the sum of the first 5 terms; and the sum of the fourth and sixth terms is 135; find the first term and the common ratio. (Princeton.)

13. A capital, c, is placed on interest at r per cent, compounded annually. What is the amount at the end of the first year? Second year? etc.

Show that at the end of the first year the amount will be $c+\frac{cr}{100}=c\left(1+\frac{r}{100}\right)$.

Show that at the end of the second year the amount will be $c\left(1+\frac{r}{100}\right)+c\left(1+\frac{r}{100}\right)\frac{r}{100}=c\left(1+\frac{r}{100}\right)\left(1+\frac{r}{100}\right)=c\left(1+\frac{r}{100}\right)^2$.

Show that at the end of the third year the amount will be $c\left(1+\frac{r}{100}\right)^3$, etc.

14. What is the amount in exercise 13 at the end of the nth year?

15. What is the amount of \$100 at 6 per cent, compounded annually, at the end of 2 years? 10 years? n years?

16. What is the amount of \$25 at $3\frac{1}{2}$ per cent, compounded annually, at the end of 6 years? 15 years?

‡**17.** A rubber ball falls from a height of 40 in. and on each rebound rises 40 per cent of the previous height. Find by formula how far it falls on its eighth descent.

‡**18.** An elastic ball drops from a height of 16 ft. on a hard pavement and rebounds again and again. If the time of each rise is $\frac{3}{4}$ of the time of the preceding fall, show that the time during which bouncing continues cannot exceed 7 seconds (the time required for the first fall is 1 second). (Harvard.)

‡**19.** A rubber ball is dropped from a height of 20 feet. After each rebound it rises to $\frac{9}{16}$ of the height from which it fell. How far does it travel before it comes to rest? (Harvard.)

218. Geometrical means. The terms of a geometrical progression included between two other terms are called **geometrical means.**

In the progression 1, 3, 9, 27 the terms 3 and 9 are two geometrical means between 1 and 27.

The following example illustrates the method of inserting a number of geometrical means between two given numbers:

Insert 3 geometrical means between 2 and 32.

Here $$a=2,\ n=5,\ l=32$$

$$\therefore 32=2r^4$$

$$\therefore r=2$$

The geometrical means are 4, 8, and 16.

EXERCISES

1. Insert three geometrical means between 486 and 6.

2. Find the geometrical mean between the two segments, Fig. 97, into which the altitude from the vertex of the right angle divides the hypotenuse.

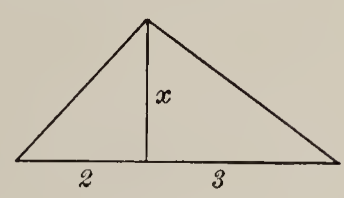


FIG. 97

3. Find the geometrical mean between 8 and 32.

4. Insert three geometrical means between $\frac{3}{4}$ and 12.

5. If $\frac{a}{b}=\frac{c}{d}$, prove that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2. (Princeton.)

MISCELLANEOUS EXERCISES

‡219. Solve the following problems:

1. Find the sum of n terms of the series

$(x-y)+\left(\frac{y^2}{x}-\frac{y^3}{x^2}\right)+\left(\frac{y^4}{x^3}-\frac{y^5}{x^4}\right)+\ldots.$ (Yale.)

2. The sum of three numbers in geometrical progression is 70. If the first be multiplied by 4, the second by 5, and the third by 4, the resulting numbers will be in arithmetical progression. Find the three numbers. (Board.)

3. If $\frac{1}{b-a}$, $\frac{1}{2b}$, and $\frac{1}{b-c}$ are in arithmetical progression, show that a, b, and c are in geometrical progression. (Yale.)

4. An arithmetical progression and a geometrical progression have the same first term, 3, equal third terms, and the difference of the second terms is 6. Determine the progressions.

5. The sum of 9 terms of an arithmetical progression is 46; the sum of the first 5 terms is 25. Find the common difference. (Vassar.)

6. The difference between two numbers is 48. Their arithmetical mean exceeds the geometrical mean by 18. Find the numbers. (Smith.)

Infinite Geometrical Series

220. Infinite geometrical series. We have learned how to find the sum of n terms of a geometrical series, n being a finite number. If the number of terms in the series $a+ar+ar^2+\ldots$ is unlimited, it is called an **infinite geometrical series.** If the sequence of the partial sums, $a,\ a+ar,\ a+ar+ar^2,\ a+ar+ar^2+ar^3$, etc., approaches a definite finite number, S, as a limit, we say S is the **sum of the infinite series**

$$a+ar+ar^2+ar^3+\ldots$$

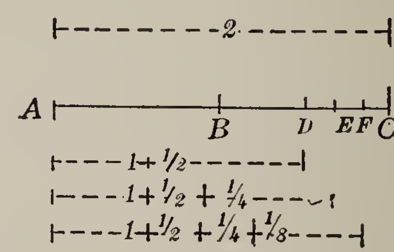


FIG. 98

For example, to find the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ we may represent the partial sums graphically, Fig. 98.

Let $AB=1=BC$. Then $AD=1+\frac{1}{2}$.

Bisecting the remainder, DC,

$$AE=1+\tfrac{1}{2}+\tfrac{1}{4}.$$

Bisecting the second remainder, EC,

$$AF=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}.$$

As the process of increasing the number of terms continues, the partial sum increases, approaching the number 2 in such a way that it can be made to differ from 2 by less than any assigned number, however small.

Hence 2 is said to be the *sum* of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\ldots$

The series is a *convergent* series.

Another example of a convergent infinite geometrical series is found in a recurring decimal fraction.

The number $.333\ldots$ is really a brief way of writing

$$\tfrac{3}{10}+\tfrac{3}{100}+\tfrac{3}{1000}+\tfrac{3}{10000}\cdots,$$

a geometrical series in which $a=\frac{3}{10}$ and $r=\frac{1}{10}$.

Show that the sum of this series is $\frac{1}{3}$.

The two preceding examples are particular cases of an infinite geometrical series whose ratio, r, is numerically *less than* 1.

If in the series $a+ar+ar^2+ar^3+\ldots.$
we take r numerically equal to 1, we have either

$$a+a+a+a+\ldots.,$$

or

$$a-a+a-a+\ldots.$$

In the first case the sum increases without bound as the number of terms increases indefinitely. In the second case the partial sums a, $a-a$, $a-a+a$, etc., have alternately the value a or 0. Hence in either case the series has no definite sum.

The question as to the sum of an infinite geometrical series may be considered by making a study of the formula for the sum of n terms,

$$s_n=a\frac{1-r^n}{1-r}=\frac{a-ar^n}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}$$

$$\therefore \quad s_n=\frac{a}{1-r}-\frac{ar^n}{1-r}$$

1. If $r>1$, then r^n increases without bound as n increases indefinitely. This is expressed symbolically by the statement $\lim\limits_{n\to\infty}(r^n)=\infty$ which is read,

"The limit of r^n, as n increases indefinitely, is infinite."

Hence s_n also increases numerically without bound and the series has no sum.

2. If $r<1$, $\lim\limits_{n\to\infty}(r^n)=0.$

For example, if $r=\frac{1}{2}$, the values of r^n for $n=1$, 2, 3, 4,.... are respectively the numbers of the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ ....; if $r=.1$, we have the sequence .1, .01, .001, .0001,....

$$\therefore \lim_{n\to\infty}\left(\frac{a}{1-r}-\frac{ar^n}{1-r}\right)=\frac{a}{1-r}.$$

Hence the infinite geometrical series $a+ar+ar^2+\ldots.$ has a sum, s, given by the formula,

$$s=\lim_{n\to\infty}(s_n)=\frac{a}{1-r}, \text{ if } r<1.$$

EXERCISES

Find the sum of each of the following infinite series:

1. $3+1+\frac{1}{3}+\frac{1}{9}+\ldots.$

Let $a=3$, $r=\frac{1}{3}$.

Then $s=\dfrac{a}{1-r}=\dfrac{3}{1-\frac{1}{3}}=\dfrac{9}{2}$.

2. $10+5+2\frac{1}{2}+1\frac{1}{4}\ldots.$

3. $-3+1-\frac{1}{3}+\frac{1}{9}\ldots.$

4. $-2-\frac{1}{4}-\frac{1}{32}\ldots.$

5. $\frac{4}{3}+\frac{2}{3}+\frac{1}{3}+\ldots.$

6. $6-4+\frac{8}{3}\ldots.$

7. $\sqrt{2}+1+\frac{1}{\sqrt{2}}+\frac{1}{2}\ldots.$

8. $1.35+0.045+0.0015+\ldots.$ (Princeton.)

9. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\ldots.$ (Harvard.)

Also find the sum of the positive terms.

10. In a geometrical progression the sum to infinity is 64 times the sum to 6 terms. What is the common ratio? (Princeton.)

11. The first term of a geometrical progression is 225 and the fourth term is $14\frac{2}{5}$. Find the series and the sum to infinity.

Find the limiting value of each of the following repeating decimals:

12. .1666. . . .

Let $a=\frac{6}{100}$, $r=\frac{1}{10}$.

Find s and add .1 to the result.

13. .363636. . . .

14. 1.2121. . . .

15. .83333. . . .

16. .234234. . . .

17. .23737

221. Historical note. The arithmetical and geometrical progressions are among the oldest topics of all mathematics. Problems leading to both kinds of progression are found in the oldest extant historical document, the papyrus of Ahmes. The forms of progression called for by the problems mentioned in this manuscript are very far from the simplest, indicating that, for thousands of years before the Christian era, Egyptian scholars had studied these progressions and by 1700 to 2000 B.C. they had attained to an advanced stage of knowledge of them.

The Babylonians made use of both forms of progression in recording the phases of the moon, and the Greeks were zealous students of the progressions. The theory was very greatly advanced by the Pythagoreans in connection with their work in *figurate numbers,* which was also a favorite subject of the school of Plato. Archimedes was even well acquainted with the laws of summation of the progressions. Heron made extended practical use of the laws. Hypsicles and Nicomachus both studied and taught the topics very fully.

The Hindus never advanced beyond the attainment of the Greeks. They solved a few problems requiring a knowledge of the progressions, and regarded the study as belonging to arithmetic. The Arabs advanced considerably beyond the Hindus. They were in possession of the completed theory of these topics, if we may judge from the work of Leonardo of Pisa, who, in his *liber abaci* of 1202 A.D., brought together what was known by the Greeks and Arabs and made it available for European scholars. Leonardo even made summing of these series one of the *nine fundamental processes* of arithmetic, thus

putting what he called *progressio* on a par with *additio, subtractio, multiplicatio,* etc.

The progressions formed a basis for the ancient Greek *method of exhaustions*, for Cavalieri's *theory of indivisibles*, see §302, for the later summation schemes of the *infinitesimal calculus*, and out of the association of an arithmetical and a geometrical series term by term grew the subject of logarithms. The topics are therefore important mathematically both for the problems they aid in solving and for the large amount of mathematical theory and methodology they have aided in developing. See Tropfke, Band II, S. 309 ff.

Summary

222. The chapter has taught the meaning of the following terms:

binomial theorem	geometrical means
arithmetical progression	infinite progression
arithmetical means	elements of a progression
geometrical progression	convergent series

223. The following are the typical problems solved in the chapter:

1. To expand a power of a binomial to a given number of terms.

2. To find a required term in the expansion of a power of a binomial.

3. Given three elements of a progression, to find the remaining two.

4. To insert a number of arithmetical, or geometrical, means between two given numbers.

5. To find the sum of n terms of a progression.

6. To find the sum of an infinite geometrical progression.

224. The following formulas have been developed.

1. $(a+b)^n = a^n + n(a)^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \ldots.$

2. $t_r = \frac{n(n-1)(n-2) \ldots (n-r+2)}{1 \cdot 2 \cdot 3 \ldots (r-1)} a^{n-r+1}b^{r-1}$

3. $l = a + (n-1)d$

4. $s = \frac{n}{2}(a+l)$

5. $l = ar^{n-1}$

6. $s = \frac{a(1-r^n)}{1-r}$

7. $s = \frac{a}{1-r}$, if $r<1$, and if the number of terms is unlimited.

CHAPTER XI

SYSTEMS OF EQUATIONS IN TWO UNKNOWNS INVOLVING QUADRATICS

Graphs of Quadratic Equations in Two Unknowns

225. General quadratic equation. The most general quadratic equation in two unknowns is

$$ax^2+bxy+cy^2+dx+ey+f=0.$$

If $b=c=0$, the equation takes the form

$$ax^2+dx+ey+f=0.$$

Solving for y,

$$y=-\frac{a}{e}x^2-\frac{d}{e}x-\frac{f}{e}.$$

This expression is of the form of the quadratic function

$$ax^2+bx+c,$$

the graph of which is known to be a *parabola*, § 13.

226. Circle. It will be seen that the parabola is not the only curve representing an equation of the second degree in two unknowns. The following examples serve as illustrations:

1. Let O, Fig. 99, be the center of a circle with radius r.

For any point on the circle, as P, the distance PO is constant and equal to r.

Let x and y be the co-ordinates of P.

By the theorem of Pythagoras,

$$x^2+y^2=r^2$$

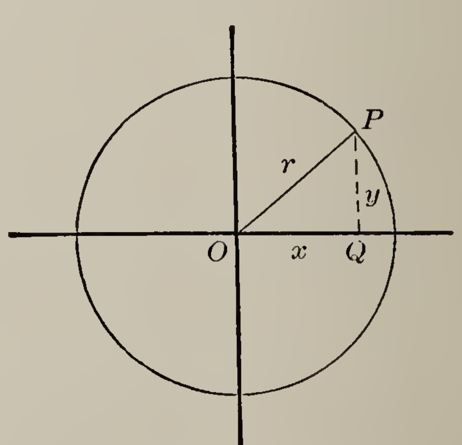


FIG. 99

This is the *equation of the circle of center O and radius r.* The graph of the equation, being a circle, is easily drawn with the compasses.

2. Let O, Fig. 100, be the center of a circle with radius r. Let the co-ordinates of O be h and k, and of P be x and y.

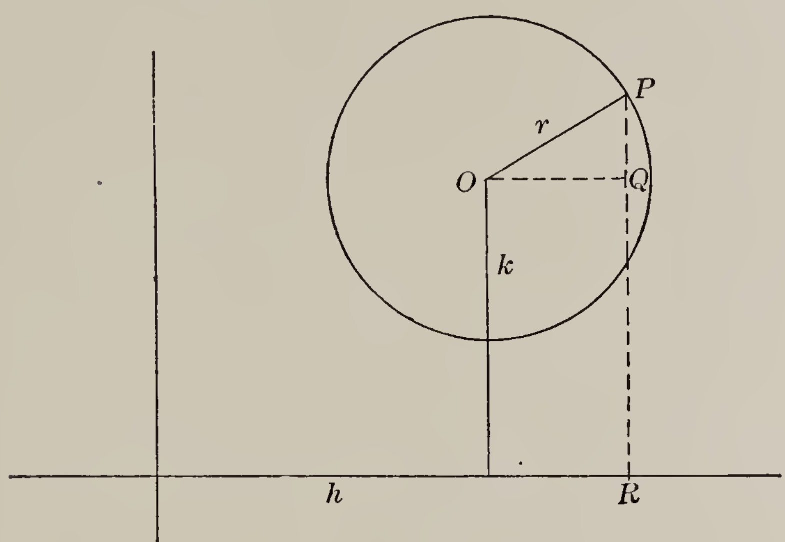


FIG. 100

Then $\overline{QP}^2+\overline{OQ}^2=r^2$.

Since $\overline{QP}=y-k$, and $\overline{OQ}=x-h$, this equation may be written

$$(x-h)^2+(y-k)^2=r^2.$$

This is the *equation of a circle* whose radius is r and whose center has the co-ordinates h and k. Expanding the squares of the binomials the equation of the circle takes the form

$$x^2+y^2-2hx-2ky+(h^2+k^2-r^2)=0. \qquad (1)$$

This is an equation of the second degree, but not of the form of the *general* equation, § 225, for it contains no xy-term, and the coefficients of x^2 and y^2 are equal. If, in the general equation, we put $b=0$, $a=c\neq 0$, show that it may be reduced to an equation of the form of equation (1).

EXERCISES

1. Show that the following are equations of circles and determine the radius of each:

$$x^2+y^2=9;\quad x^2+y^2=5;\quad 2x^2+2y^2=8.$$

2. Give the equations of circles having the center at the origin and a radius equal to 16; a; $\sqrt{2}$.

3. Compare the equations in exercise 1 with the general quadratic in § 225 and in each case determine the values of the coefficients.

227. Graph of the equation

$$x^2+y^2-2hx-2ky+(h^2+k^2-r^2)=0.$$

If particular values are assigned to h and k and r, as $h=4$, $k=3$, $r=2$, this equation reduces to

$$x^2+y^2-8x-6y+21=0,$$

or

$$y^2-6y+(x^2-8x+21)=0.$$

1. Solving for y,

$$y=\frac{6\pm\sqrt{-4x^2+32x-12}}{2}=3\pm\sqrt{-x^2+8x-3}.$$

2. The table, Fig. 101, gives the values of y corresponding to assumed values of x.

x	y
0	$3\pm\sqrt{-3}$, imaginary
1	3 ± 2, or 5, 1
2	3 ± 3, or 6, 0
3	$3\pm\sqrt{12}$, or 6.5, $-.5$
4	$3\pm\sqrt{13}$, or 6.6, $-.6$
5	$3\pm\sqrt{12}$, or 6.5, $-.5$
6	3 ± 3, or 6, 0
7	3 ± 2, or 5, 1
8	$3\pm\sqrt{-3}$, imaginary

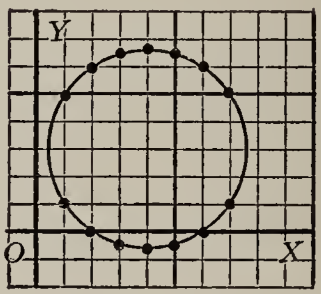


FIG. 101

3. By plotting the points determined by the values in Fig. 101, we obtain the required graph.


Hieronimus Cardanus 59

GIROLAMO CARDANO

GIROLAMO CARDANO (English, Cardan) was born at Pavia in 1501 and died at Rome in 1576. That Cardan was one of the leading algebraists of all time there is no question. The chief mathematical work of Cardan is the *Ars magna*, published in 1545. It was both epoch-making and a masterpiece of algebraic scholarship. It gave to the world for the first time the solution of both the cubic and the biquadratic equations.

That Cardan discovered neither of these solutions there is no doubt. Cardan never claimed them as his discoveries. The credit of discovery of the solution of the cubic that is commonly referred to in American texts as Cardan's solution, is quite generally regarded as due to Tartaglia. The facts seem to be that one Scipio Ferro in some unknown way came into possession of the solution of the form $x^3+ax=b$, and told it to numerous friends, among them one Fiori (Floridas). According to Cardan this occurred in 1515 and according to Tartaglia as early as 1506. In 1535 Fiori used this knowledge to propose, in accord with a custom of the day, thirty challenge problems to Tartaglia, then a mathematical teacher of repute. Tartaglia knew well that all the problems would lead to the form $x^3+ax=b$ and he bestirred himself for a long time vainly to discover the solution; but eight days before the contest he says he succeeded. The next day he discovered the solution formula for $x^3=ax+b$. In the contest he solved all of Fiori's problems in some two hours.

Cardan, then a professor of mathematics at Bologna, besought Tartaglia to publish his solution. Tartaglia refused. Later in obscure verse he did hint at it. . Cardan agreed not to divulge Tartaglia's solution if Tartaglia would impart it to him. A letter of Tartaglia's later explained, again obscurely, some things hinted at in the verse. Then in 1545 when Cardan's *Ars magna* appeared it contained the correct solution with proof, of the cubic and the biquadratic equations, with due credit given both to Tartaglia and to Ferari, who was discoverer of the solution of the biquadratic. Tartaglia was now greatly annoyed and accused Cardan of violating a solemn pledge of secrecy. Later Cardan showed that in the literary remains of Professor Ferro was the solution of the cubic precisely identical with that of Tartaglia and dating thirty years earlier than Tartaglia's contest.

Debate: Is Cardan justly chargeable with literary theft for publishing Tartaglia's solution which was identical with Ferro's of thirty years earlier date?

[See Tropfke, *Geschichte der Elementar-Mathematik*, Band I, S. 274 ff.]

EXERCISES

1. Graph the equation

$$x^2+y^2+6x-16=0.$$

2. Graph the equation

$$(x-7)^2+(y+8)^2=36.$$

228. The ellipse. If $b=d=e=0$, the general quadratic, § 225, reduces to

$$\boldsymbol{ax^2+cy^2+f=0.} \qquad (1)$$

Show that the equation $16x^2+25y^2=400$ is of this form. Dividing both sides of this equation by 400,

$$\frac{x^2}{25}+\frac{y^2}{16}=1.$$

This equation is of the form

$$\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.} \qquad (2)$$

Equation (1) is better adapted to computing the corresponding values of x and y than equation (2).

The equation $\frac{x^2}{25}+\frac{y^2}{16}=1$ may be graphed as follows:

1. Solve for y,

$$y=\pm\frac{4}{5}\sqrt{25-x^2}=\pm(.8)\sqrt{25-x^2}.$$

2. Obtain the corresponding values of x and y as given in the table, Fig. 102.

x	y
0	± 4
1	± 3.9
2	± 3.7
3	± 3.2
4	± 2.4
5	± 0

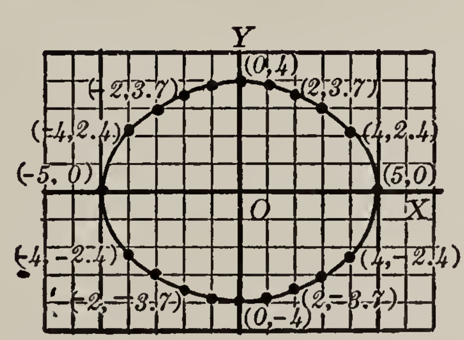


FIG. 102

3. Graph these values and draw the curve through the points thus obtained.

The curve in Fig. 102 is an **ellipse.**

Equation (2) readily shows:

1. That the curve is symmetric with respect to the axes.

2. That the intercepts on the axes are $\pm a$ and $\pm b$.

The lengths a and b are the *semiaxes* of the ellipse. If $a=b$, the ellipse reduces to a circle.

EXERCISES

1. Graph the equation

$$4x^2+y^2=20.$$

2. Graph the equation

$$x^2+4y^2=16.$$

229. The Hyperbola. If in the equation

$$ax^2+cy^2+f=0,$$

c is negative and a positive, the equation may be changed to the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

For $a=5$ and $b=4$, this reduces to

$$\frac{x^2}{25}-\frac{y^2}{16}=1.$$

The equation may be graphed as follows:

1. Solving for y,

$$y=\pm(.8)\sqrt{x^2-25}.$$

2. The corresponding values of x and y are computed and tabulated as in Fig. 103.

x	y
0	$\pm(.8)\sqrt{-25}$, imaginary
± 5	0
± 6	± 2.6
± 7	± 3.9
± 8	$\pm 5.$
± 9	$\pm 6.$
$\vdots$	$\vdots$

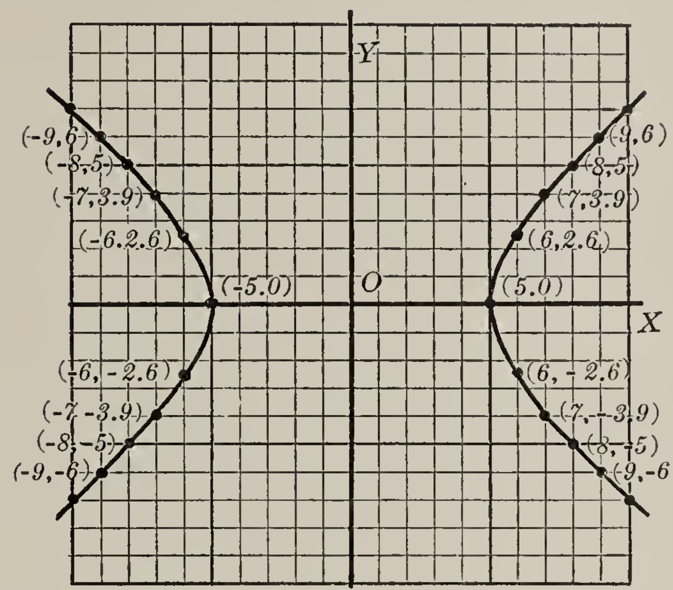


FIG. 103

3. The curve, Fig. 103, is the graph of the equation $\frac{x^2}{25}-\frac{y^2}{16}=1.$

The curve is called a **hyperbola.**

EXERCISES

1. Graph the equation

$$x^2-y^2-4=0.$$

2. Graph the equation

$$\frac{x^2}{9}-\frac{y^2}{4}=1.$$

230. The graph of an equation of the form

$$xy=c$$

is also a hyperbola. This equation was discussed in § 24.

If $c=8$, the graph $xy=8$, Fig. 104, is easily obtained by means of the values in the table, Fig. 104.

x	y
1	+8
2	+4
3	+2.7
4	+2
5	+1.6
6	+1.3
7	+1.1
8	+1
−1	−8
−2	−4
−3	−2.7
−4	−2
−5	−1.6
−6	−1.3
−7	−1.1
−8	−1
⋮	⋮

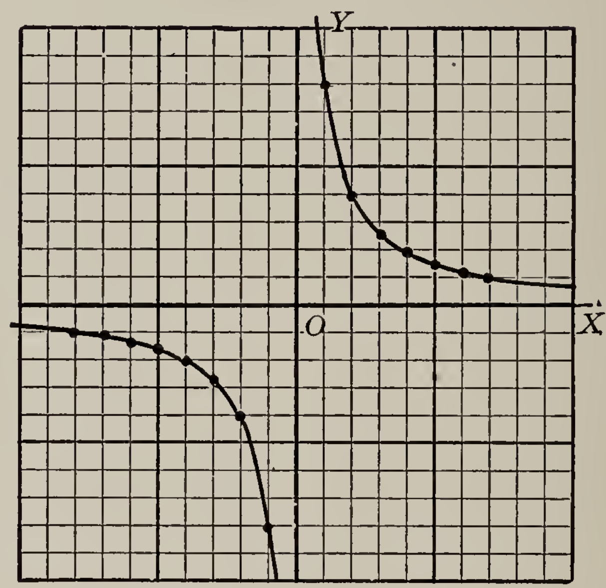


FIG. 104

EXERCISES

1. Graph the equation

$$xy=24.$$

2. Graph the equation

$$xy=5.$$

231. Two straight lines. If the general quadratic equation is the product of two linear factors, the graph consists of two *straight lines*, as illustrated in the following example:

$$x^2+2xy+y^2+2x+2y-3=0.$$

By grouping terms,

$$(x+y)^2+2(x+y)-3=0$$

$$\therefore\ (x+y+3)(x+y-1)=0.$$

The graph consists of two parallel straight lines, Fig. 105.

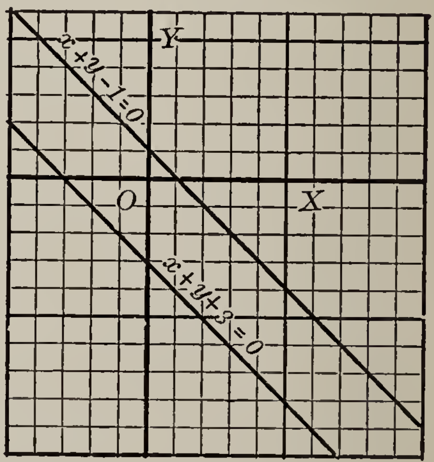


FIG. 105

Solution of Simultaneous Quadratics

232. To solve a *system of quadratic equations* in two unknowns, as

$$\begin{cases} y-x^2=2 \\ x-y^2=5 \end{cases}$$

one of the unknowns may be eliminated by *substitution.*

Since $y=2+x^2$, the second equation is changed to

$$x-(2+x^2)^2=5,$$

or

$$x-4-4x^2-x^4=5,$$

$$\therefore \quad x^4+4x^2-x+9=0.$$

This is an equation of the fourth degree. So far the student knows no *general* method of solving an equation of the fourth degree and therefore he will find it impossible to solve some systems of simultaneous quadratics.

A similar situation was found in the study of factoring polynomials. Not being able to work out a general method by which *any* polynomial may be factored, we made a study of certain typical forms of polynomials. Methods were then worked out for factoring such special forms.

Similarly we shall now study only certain cases of simultaneous quadratics and find the proper methods of solution.

233. Case I. *The form of the equations is such that either x or y may be eliminated by addition or subtraction.* The following example illustrates case I:

EXERCISES

Solve the system of equations

$$\begin{cases} x^2+\ y^2=\ 25 \\ 4x^2+9y^2=144 \end{cases}$$

1. *Graphical solution.*—The graph of the first equation is a circle, § 226. The graph of the second equation is an ellipse, § 228.

The two graphs intersect in four points, Fig. 106, the co-ordinates of which must satisfy both equations. Hence there are *four solutions:*

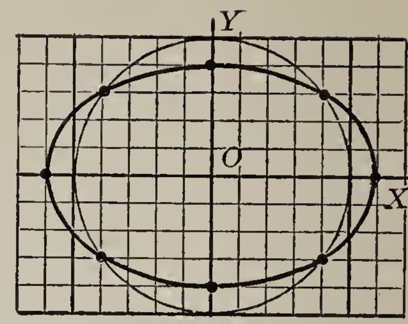


Fig. 106

(4.1, 2.9), (4.1, −2.9), (−4.1, 2.9) and (−4.1, −2.9).

2. *Algebraic solution.*—Multiplying the first equation of the system by 4 and leaving the second equation as it is, we have

$$4x^2+4y^2=100$$

and
$$4x^2+9y^2=144$$

By subtracting,
$$5y^2=44$$

$$\therefore\ y_1=\sqrt{8.8},\ y_2=-\sqrt{8.8}.$$

Substituting the value of y_1 in the first of the given equations,

$$x^2+8.8=25$$
$$\therefore\ x^2=16.2$$
$$\therefore\ x=\pm\sqrt{16.2}.$$

Hence we have the two following solutions:

$$(\sqrt{16.2},\ \sqrt{8.8}) \text{ and } (-\sqrt{16.2},\ \sqrt{8.8}).$$

Similarly, by substituting $y_2=-\sqrt{8.8}$ in the first of the given equations, we have the solutions:

$$(\sqrt{16.2},\ -\sqrt{8.8}) \text{ and } (-\sqrt{16.2},\ -\sqrt{8.8}).$$

Verify the four solutions by means of the graph, Fig. 106.

Solve the following systems:

1. $\begin{cases}5x^2-2y^2=30\\2x^2+y^2=57\end{cases}$

2. $\begin{cases}2x^2+y^2-33=0\\x^2+2y^2-54=0\end{cases}$

‡3. $\begin{cases}x^2+y^2=16\\4x^2-9y^2=36\end{cases}$

4. $\begin{cases}\dfrac{x^2}{2}+\dfrac{xy}{3}=\dfrac{8}{3}\\[2mm]\dfrac{x^2}{3}-\dfrac{xy}{4}=\dfrac{5}{6}\end{cases}$

‡5. $\begin{cases}x^2+y^2=1\\x^2-y^2=1\end{cases}$

6. $\begin{cases}4x^2-9y^2=36\\x^2+4y^2=4\end{cases}$

7. $\begin{cases}9x^2+4y^2=36\\9x^2+16y=33\end{cases}$

‡8. $\begin{cases}x^2+y^2=25\\x^2-3y=21\end{cases}$

234. Case II. *One equation can be resolved into two linear factors.* The following example illustrates case II:

Solve the following system of equations:

$$\begin{cases}4y^2-3x^2+xy+15x-20y=0\\ x^2+\ y^2=25\end{cases}$$

1. *Graphical solution.*—By factoring, the first equation is changed to the form

$$(4y-3x)(x+y-5)=0.$$

Hence the graph consists of two straight lines, Fig. 107.

The graph of $x^2+y^2=25$ is a circle, § 226.

∴ the co-ordinates of the four points of intersection give the solutions of the system.

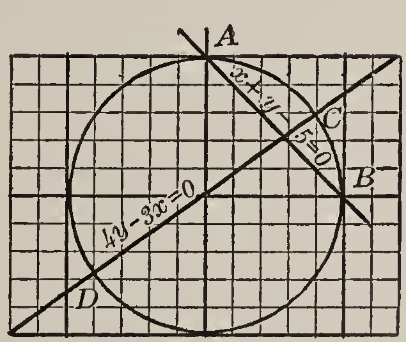


FIG. 107

2. *Algebraic solution.*—According to the graph, Fig. 107, *two* of the solutions of the system are obtained from the points of intersection, A and B, of the circle with the straight line whose equation is $x+y-5=0$.

The *other two* solutions are given by the points of intersection, C and D, of the circle with the straight line whose equation is $4y-3x=0$.

This suggests that the solutions of the original system be obtained by solving the systems:

$$\begin{cases} x+y-5=0 \\ x^2+y^2=25 \end{cases} \text{ and } \begin{cases} 4y-3x=0 \\ x^2+y^2=25 \end{cases}$$

Both systems should be solved by *eliminating* one of the unknowns *by substitution.*

Solving the first system, we have:

$$x=5-y.$$

Substituting, $(5-y)^2+y^2=25,$

$$\therefore\ 2y^2-10y=0,$$

$$\therefore\ y_1=0,\ y_2=5.$$

The corresponding values of x are found by substituting these values of y into the equation

$$x=5-y.$$

This gives the solutions: (5, 0) and (0, 5).

Find the remaining solutions.

EXERCISES

Solve the following systems:

1. $\begin{cases} x^2-5xy+6y^2=0 \\ x^2-y^2=27 \end{cases}$

2. $\begin{cases} x^2-y^2=0 \\ x^2+y^2=8 \end{cases}$

3. $\begin{cases} x^2+xy=0 \\ x^2-xy+y^2=27 \end{cases}$

‡**4.** $\begin{cases} x^2=y^2 \\ 3x^2+5y^2=32 \end{cases}$

‡**5.** $\begin{cases} 4x^2-9y^2=0 \\ 4x^2+9y^2-1=0 \end{cases}$

‡**6.** $\begin{cases} x^2-3xy=0 \\ 5x^2+3y^2-9=0 \end{cases}$

7. $\begin{cases} x^2+2xy+y^2+x+y-30=0 \\ xy=15 \end{cases}$

8. $\begin{cases} m^2+n^2-5m-5n=-4 \\ mn=5 \end{cases}$

Multiply the second equation by 2 and add to the first equation.

235. Case III. *One equation is of the form* $xy=c$. The following example illustrates this case:

Solve the system

$$\begin{cases} x^2+y^2=5 \\ xy=2 \end{cases}$$

1. *Graphical solution.*—The graph of the first equation is a circle whose radius is 2.2, approximately, Fig. 108.

The graph of the second equation is a hyperbola.

The two curves intersect in points A, B, C, and D, which determine the solutions.

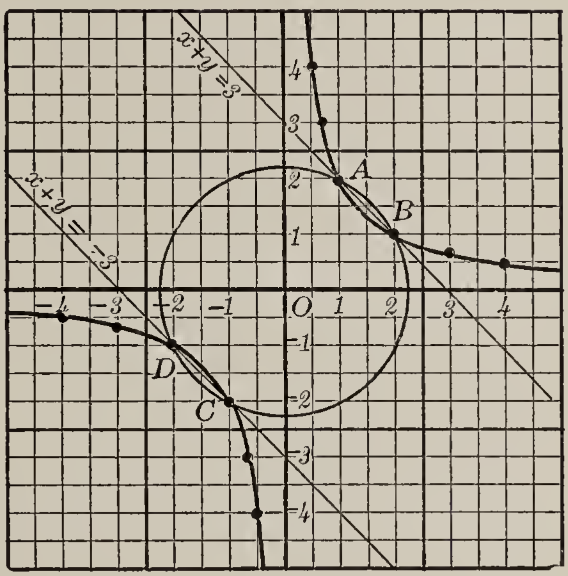


Fig. 108

2. *Algebraic solution.*—Multiply the second equation of the given system by 2, and add the resulting equation to the first. This gives the equation

$$x^2+2xy+y^2=9,$$

in which the left number is a perfect square.

Extracting the square root of both members, we obtain the two linear equations

$$x+y=+3.$$

and

$$x+y=-3.$$

The graphs of these two equations must pass through the four points of intersection of the hyperbola and circle, Fig. 108.

Therefore two of the four required points may be located by graphing the straight line $x+y=3$ and the simpler of the given equations, $xy=2$.

The other two points may be determined by the graphs of $x+y=-3$ and $xy=2$.

This suggests that, instead of solving the original system, we may solve the two systems,

$$\begin{cases} x+y=3 \\ xy=2 \end{cases} \text{ and } \begin{cases} x+y=-3 \\ xy=2 \end{cases}$$

Substituting for x its equal $3-y$,

$$(3-y)y=2,$$

$$\therefore \quad y^2-3y+2=0.$$

From this we find the solutions:

$$(1, 2) \text{ and } (2, 1).$$

Substituting for x its equal $-3-y$,

$$(-3-y)y=2,$$

$$y^2+3y+2=0.$$

Show that the solutions are:

$$(-2, -1) \text{ and } (-1, -2).$$

Often, when neither of the given equations is of the form $xy=c$, we can obtain linear equations by addition or subtraction, and extraction of the square root or factoring.

Thus, in exercise 2 below, the two given equations may be added. The resulting equation may be then divided by 2. The two linear equations are obtained by extracting the square root of both members of the last equation.

In exercise 3 the two given equations may be added. By factoring both members of the resulting equation, we obtain two linear equations.

EXERCISES

Solve the following systems:

1. $\begin{cases} x^2+y^2=13 \\ xy=6 \end{cases}$

2. $\begin{cases} x^2-5xy+y^2=-2 \\ x^2+9xy+y^2=34 \end{cases}$

3. $\begin{cases} x^2+xy+y^2=7 \\ x+xy+y=5 \end{cases}$

4. $\begin{cases} r^2+rs=5 \\ rs+s^2=4 \end{cases}$

5. $\begin{cases} x^2+y^2+x+y=18 \\ xy=6 \end{cases}$

6. $\begin{cases} x^2+xy=6 \\ xy+y^2=10 \end{cases}$

7. $\begin{cases} t^2+u^2=5 \\ tu+t+u=5 \end{cases}$

8. $\begin{cases} r^2+s^2+r+s=8 \\ rs=2 \end{cases}$

236. Case IV. *All terms containing the unknowns are of the second degree.* If in an equation all terms containing the unknowns are of the same degree the equation is **homogeneous** with respect to these terms. The following example illustrates case IV:

Solve the following system of equations:

$$\begin{cases} x^2+xy+y^2=7 \\ x^2-xy+y^2=19 \end{cases}$$

Multiplying the first equation by 19, the second by 7,

$$19x^2+19xy+19y^2=19\cdot 7$$

and

$$7x^2-7xy+7y^2=19\cdot 7.$$

Subtracting one of these equations from the other, we have

$$12x^2+26xy+6y^2=0.$$

Dividing by 2, $\quad 6x^2+13xy+6y^2=0.$

Factoring, $\quad (2x+3y)(3x+2y)=0.$

$\therefore$ we can replace the given system by the following:

$$\begin{cases} 2x+3y=0 \\ x^2+xy+y^2=7 \end{cases} \quad \text{and} \quad \begin{cases} 3x+2y=0 \\ x^2+xy+y^2=7 \end{cases}$$

Solve these two systems.

EXERCISES

Solve the following systems:

1. $\begin{cases} x^2+3xy-y^2=3 \\ 2x^2+5xy+y^2=8 \end{cases}$

2. $\begin{cases} x^2+xy=75 \\ y^2+x^2=125 \end{cases}$

3. $\begin{cases} x^2+2xy+2y^2=10 \\ 3x^2-xy-y^2=51 \end{cases}$

4. $\begin{cases} 4x^2-2xy=y^2-16 \\ 5x^2-7xy+36=0 \end{cases}$

5. $\begin{cases} x^2+3xy=7 \\ x^2-xy+y^2=3 \end{cases}$

6. $\begin{cases} x^2-xy+y^2=21 \\ y^2-2xy=-15 \end{cases}$

Solution of Equations of Degree Higher than the Second

237. One equation is divisible by the other. Some systems of quadratic equations, and of equations of higher degree may be solved by dividing one equation by the other, as in the following example:

Solve the system of equations:

$$\begin{cases} x^3+y^3=9 & (1) \\ x+y\ =3 & (2) \end{cases}$$

Since $x+y$ is a non-zero constant, we may divide the first equation by the second without losing a solution of the given system.

This gives

$$x^2-xy+y^2=3. \qquad (3)$$

Graphical solution.—Solving equation (1) for y,

$$y=\sqrt[3]{9-x^3}.$$

By means of the table of cube roots, verify the corresponding values of x and y, as given in table (1), Fig. 109.

x	y
0	2.1
1	2
2	1
3	−2.6
4	−3.8
−1	2.1
−2	2.6
−3	3.3
−4	3.8

(1)

x	y
0	±1.7
1	2, −1
2	1, 1
3	Imaginary
−1	−1, −2
−2	−1, −1
−3	Imaginary

(3)

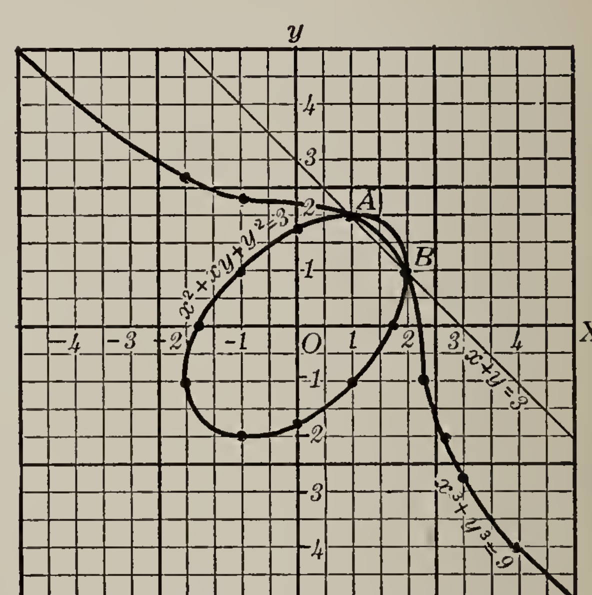


FIG. 109

Graph equations (1) and (2) and give the solution of the given system.

Solving equation (3),

$$y=\frac{x\pm\sqrt{12-3x^2}}{2}.$$

By means of this equation obtain the values in table (3), Fig. 109.

It is seen that the graph of equation (3) passes through the points of intersection of the graphs of equations (1) and (2).

Hence, by solving the system

$$\begin{cases} x^2-xy+y^2=3 \\ x+y=3, \end{cases}$$

we are able to find the required solutions of the given system.

Give the complete algebraic solution.

EXERCISES

Solve the following systems:

1. $\begin{cases} a^2-b^2=3 \\ a-b=1 \end{cases}$

2. $\begin{cases} a^3+b^3=18 \\ a+b=6 \end{cases}$

3. $\begin{cases} x^3+y^3=27 \\ x+y=3 \end{cases}$

4. $\begin{cases} a^2b+ab^2=126 \\ a+b=9 \end{cases}$

5. $\begin{cases} s^2-t^2=228 \\ st-t^2=42 \end{cases}$

6. $\begin{cases} r^3-s^3=56 \\ r^2+rs+s^2=28 \end{cases}$

7. $\begin{cases} a^4+a^2b^2+b^4=91 \\ a^2+ab+b^2=13 \end{cases}$

8. $\begin{cases} a^2+ab+2b^2=74 \\ 2a^2+2ab+b^2=73 \end{cases}$

Solution of Irrational and Fractional Equations

238. Introduction of a new variable. Some equations in which the unknowns appear in combinations may be simplified by using a new symbol in place of these combinations. Thus the equations

$$(x+y)^2+3(x+y)=-2,\ x+y+\sqrt{x+y}=6, \text{ and } \frac{1}{a^2}+\frac{1}{b^2}=8,$$

may be written respectively

$$a^2+3a=-2,\ a^2+a=6, \text{ and } x^2+y^2=8.$$

This device may be used in the exercises below.

EXERCISES

Solve the following systems:

1. $\begin{cases} a+b+\sqrt{a+b}=6 \\ a^2+b^2=10 \end{cases}$

Denote $\sqrt{a+b}$ by x. Solve the first equation for x and obtain two linear equations in a and b.

2. $\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{3} \\ \dfrac{1}{x^2}-\dfrac{1}{y^2}=\dfrac{1}{4} \end{cases}$

Put $\dfrac{1}{x}=a$, $\dfrac{1}{y}=b$.

3. $\begin{cases} x+y=72 \\ \sqrt[3]{x}+\sqrt[3]{y}=6 \end{cases}$

Let $\sqrt[3]{x}=a$, $\sqrt[3]{y}=b$.

4. $\begin{cases} \dfrac{1}{x^2}-\dfrac{1}{xy}+\dfrac{1}{y^2}=7 \\ \dfrac{1}{x}+\dfrac{1}{y}=4 \end{cases}$

5. $\begin{cases} r+s-r^2s^2=\dfrac{1}{2} \\ r^2+s^2-r^4s^4=\dfrac{1}{4} \end{cases}$

6. $\begin{cases} ab+5a+5b=23 \\ 4ab+3a+3b=2ab(a+b) \end{cases}$

MISCELLANEOUS EXERCISES

239. Solve the following problems:

1. The perimeter of a rectangle is 22 inches. If the cube of its length is added to the cube of its width the result is 407. Find the area of the rectangle.

2. The difference of the cubes of two consecutive numbers is 817. Find the numbers. (Chicago.)

3. The sum of two numbers multiplied by the greater is 126 and their difference multiplied by the less is 20. Find the numbers. (Princeton.)

4. The area of a rhombus is 24 sq. in. and the sum of its diagonals is 14 inches. Find the length of one side. (Harvard.)

‡**5.** The sum of the volumes of two cubes is 559 cu. in. and the sum of their lengths is 13 inches. What is the height of each cube?

6. At his usual rate a man can row 15 mi. downstream in 5 hr. less time than it takes him to return. Could he double his rate, his time downstream would be 2 hr. less than his time upstream. What is his usual rate in still water and what is the rate of the current? (Board.)

‡**7.** The diagonal of a rectangle is 13 ft. long. If each side were longer by 2 ft., the area would be increased by 38 square feet. Find the length of the sides.

8. Two men, A and B, start at the same time from a certain point and walk east and south respectively. At the end of 5 hr. A has walked 5 mi. farther than B, and they are 25 mi. apart. Find the rate of each.

9. If a number of two digits is divided by the sum of the digits the quotient is 2 and the remainder is 2. If it is multiplied by the sum of the digits, the product is 112. Find the number. (Board.)

10. Three men, A, B, and C, can do a piece of work together in 1 hr. and 20 minutes. To do the work alone C would take twice as long as A and 2 hr. longer than B. How long would it take each to do the work alone? (Board.)

11. Find two numbers such that their sum, difference, and the sum of their squares are in the ratio 5:3:51. (Yale.)

12. The sum of the ages of a father and his son is 100 years, and one-tenth of the product of the numbers of years in their ages minus 180 equals the number of years in the father's age. What is the age of each?

13. Solve the equations

$$\begin{cases} l=a+(n-1)d \\ 2s=n(a+l) \end{cases}$$

taking l and n as the two unknown numbers. Find l and n when $a=\frac{1}{3}$, $d=-\frac{1}{12}$, $s=-\frac{3}{2}$. (Princeton.)

14. Two men, A and B, dig a trench in 20 days. It would take A alone 9 days longer to dig it than it would B. How long would it take A and B each working alone? (Yale.)

15. Two automobiles run 336 miles. The winning car wins by 4 hr. by going 2 mi. an hour faster than the other. What was the winner's time and speed? (Sheffield.)

16. Two men work on a job and each receives 36 dollars. One of them, however, has worked 2 days less than the other and is paid 20 cents more a day. Find his daily wages and the number of days he worked. (Sheffield.)

17. An audience of 360 persons is seated in rows each containing the same number of people. They might have been seated in four rows less if each row contained 3 more people. How many rows were there? (Board.)

18. Find the sides of a rectangle whose area is unchanged if its length is increased by 4 ft. and its breadth decreased by 3 ft., but which loses one-third of its area if the length is increased by 16 ft. and the breadth decreased by 10 feet. (M.I.T.)

Solve the following systems:

19. $\begin{cases} x^2-3xy-y^2=9 \\ 2x^2+2xy+3y^2=7 \end{cases}$ (Board)

20. $\begin{cases} s^2+st+s-t=-2 \\ 2s^2-st-t^2=0 \end{cases}$

21. $\begin{cases} x+y=5-xy \\ x+y=\dfrac{6}{xy} \end{cases}$ (Yale)

22. $\begin{cases} x^2+xy+y^2=133 \\ x-\sqrt{xy}+y=7 \end{cases}$

23. $\begin{cases} 3xy=1 \\ 36x^2+180xy+36y^2=-35 \end{cases}$ (Sheffield)

24. $\begin{cases} x-y=1 \\ \dfrac{x}{y}-\dfrac{y}{x}=\dfrac{5}{6} \end{cases}$

25. $\begin{cases} x^2-xy=3 \\ xy-y^2=2 \end{cases}$

26. $\begin{cases} x^2-4xy+4y^2-x+2y-6=0 \\ 4x^2+12xy+9y^2+2x+3y-12=0 \end{cases}$

27. $\begin{cases} x+\dfrac{1}{y}=1 \\ y+\dfrac{1}{x}=4 \end{cases}$ (Harvard)

28. $\begin{cases} s^2+st+t^2=13s \\ s^2-st+t^2=7s \end{cases}$ (Chicago)

29. $\begin{cases} 5x^2y^2-2=3xy \\ x+5y=1 \end{cases}$ (Princeton)

30. $\begin{cases} \sqrt{x}-\sqrt{y}=2 \\ (\sqrt{x}-\sqrt{y})(\sqrt{xy})=30 \end{cases}$ (Yale)

31. $\begin{cases} 2x+\sqrt{xy}=10 \\ 3y-2\sqrt{xy}=-1 \end{cases}$

32. $\begin{cases} xy=80 \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{5} \end{cases}$

33. $\begin{cases} x^2y^2+28xy-480=0 \\ 2x+y=11 \end{cases}$ (Yale)

34. $\begin{cases} x^2+xy+y^2=21 \\ x-\sqrt{xy}+y=3 \end{cases}$

Summary

240. The chapter has taught how to graph the following quadratic equations:

1. The equation $y=ax^2+bx+c$, representing a parabola.

2. The equations $x^2+y^2=a^2$ and $x^2+y^2+ax+by+c=0$ representing circles.

3. The graph of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, or $b^2x^2+a^2y^2=a^2b^2$ representing an ellipse.

4. The equations $xy=c$ and $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ representing hyperbolas.

241. Outline the methods of solving the following systems of equations:

1. $\begin{cases} x+y-2=0 \\ 4x^2-2x+5=y \end{cases}$

2. $\begin{cases} x^2+y^2=25 \\ 4x^2+9y^2=144 \end{cases}$

3. $\begin{cases} x^2+2xy+y^2+x+y-12=0 \\ x^2+y^2=36 \end{cases}$

4. $\begin{cases} x^2+y^2=49 \\ xy=24 \end{cases}$

5. $\begin{cases} x^2+y^2+xy=4 \\ x^2+2y^2-xy=7 \end{cases}$

6. $\begin{cases} x^3-y^3=21 \\ x-y=3 \end{cases}$

7. $\begin{cases} x+y-x^2y^2=2 \\ x^2+y^2-x^4y^4=4 \end{cases}$

CHAPTER XII

AREAS OF SURFACES

242. In the world about us some forms of material objects occur frequently, e.g., the prismatic forms of boxes, houses, and posts; the cylindrical forms of boilers, pipes, columns, and tanks; the conical forms of funnels, pails, and spires; and the spherical forms of globes, domes, and balls. It is the purpose of this chapter to study the surfaces of these forms and to develop formulas by means of which we may compute the areas of these surfaces.

Polyedrons. Cylinders. Cones

243. Polyedron. A geometrical solid bounded entirely by planes is a **polyedron.***

EXERCISES

1. Give some examples of polyedrons.

2. What is the least number of planes necessary to form a polyedron?

244. Face. Surface. Edge. Vertices. The boundary planes of the polyedron are the **faces**; the sum of the faces is the **surface**; the intersections of the faces are the **edges**; the intersections of the edges are the **vertices.**

245. Classification of polyedrons. Polyedrons may be classified according to the number of faces. Thus,

* Some authors write *polyhedron* instead of *polyedron.*

Fig. 110, a polyedron of *four* faces is a *tetraedron;* of *six*

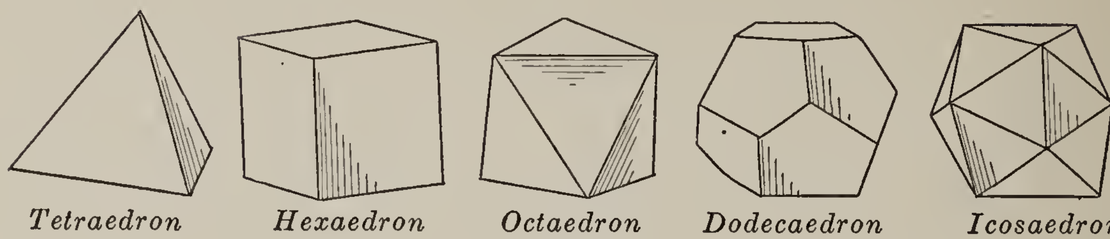


FIG. 110

faces a *hexaedron;* of *eight* faces an *octaedron;* of *twelve* faces a *dodecaedron;* of *twenty* faces an *icosaedron.*

246. Relation between the number of faces, vertices, and edges of a convex polyedron. Count the number of faces, vertices, and edges in each of the figures in § 245 and tabulate the results as below:

Number of faces	4	6	8	12
Number of vertices	4	8	6	20
Sum	8	14	14	32
Number of edges	6	12	12	30

Denoting the number of faces, vertices, and edges by f, v, and e, respectively, compare e with $f+v$. State the relation between e and $f+v$ in the form of an equation.

The mathematician Leonard Euler (1707–83) *proved* that $\mathbf{e+2=f+v}$.

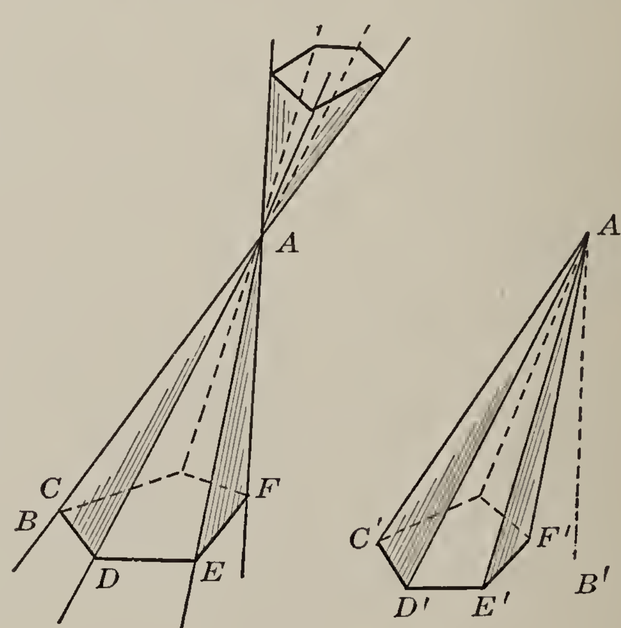


FIG. 111

247. Pyramid. If a line, AB, Fig. 111, pass-

ing through a fixed point, A, moves always touching a convex polygon, $CDEF$, whose plane does not contain A, it generates a *pyramidal surface.* The generating line AB is called the *generatrix,* the polygon $CDEF$ the *directrix,* and the fixed point A the *vertex.*

The solid $A'-C'D'E'F'$ bounded by the pyramidal surface and by the plane of the polygon is a **pyramid.**

The polygon $C'D'E'F'$ is the *base.*

The portion of the surface between the vertex and the base is the *lateral surface.*

The perpendicular distance $A'B'$ from the vertex to the plane of the base is the *altitude.*

The edges $A'C'$, $A'D'$, etc., are the *lateral edges* of the pyramid.

248. Cone. If a line, Fig. 112, passing through a fixed point, moves always touching a convex closed curve whose plane does not contain the fixed point, it generates a surface called a *conical surface.* The generating line is the *generatrix,* the curve the *directrix,* and the fixed point the *vertex.* The generatrix in any position is called an *element.*

The solid bounded by the conical surface and the plane of the curve is a **cone.**

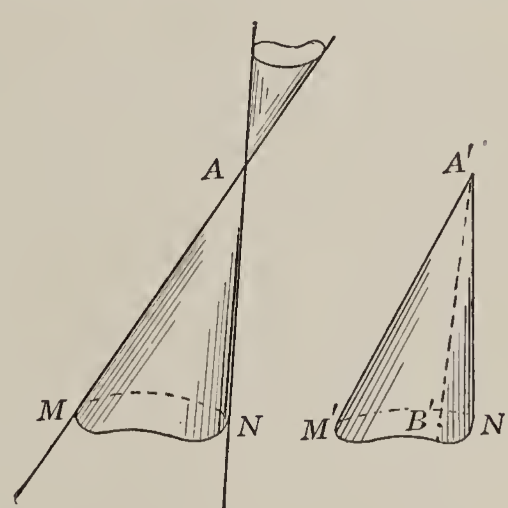


Fig. 112

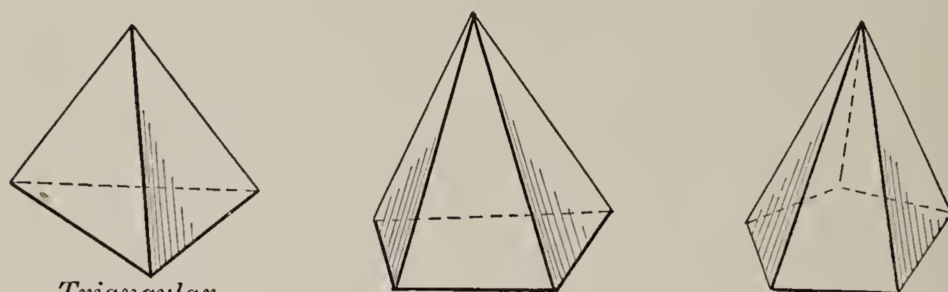
The curve cut from the plane by the conical surface is the *base.*

The portion of the surface between the vertex and the base is the *lateral surface.*

The perpendicular distance from the vertex to the plane of the base is the *altitude.*

The two parts of the pyramidal, or conical, surface on opposite sides of the fixed points are called the *nappes*.

249. Classification of pyramids. Pyramids are *triangular, quadrangular, pentagonal,* etc., according as the bases are triangles, quadrilaterals, pentagons, etc., Fig. 113.



Triangular Pyramid Tetraedron *Quadrangular Pyramid* *Pentagonal Pyramid*

FIG. 113

A triangular pyramid is a **tetraedron,** Fig. 113. Any one of the four triangular faces of a tetraedron may be taken as the base.

If the base of a pyramid is a regular polygon, Fig. 114, and if the altitude, VH, meets the base at the center of the circumscribed circle, the pyramid is said to be **regular.**

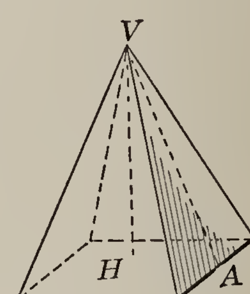


FIG. 114

The altitude VA, of one of the triangular faces of a regular pyramid, is the **slant height.**

EXERCISES

1. Give examples of material objects of the form of a pyramid.

2. Prove that *the lateral edges of a regular pyramid are equal.*

3. Prove that *the lateral faces of a regular pyramid are congruent isosceles triangles.*

4. Prove that the slant height of a regular pyramid is the same for all lateral faces.

5. The lines joining the midpoints of four edges of a tetraedron, no three of which pass through the same vertex, form a parallelogram.

6. Prove that the lines connecting the middle points of the opposite edges of a tetraedron bisect one another. (Harvard.)

250. Classification of cones. A cone whose base is a circle is a **circular cone,** Fig. 115. Only *circular* cones are studied in this chapter. The line joining the vertex, V, to the center, C, of the base is the **axis** of the cone. A circular cone whose axis is perpendicular to the base is a **right circular cone.** If the axis is oblique to the base, the

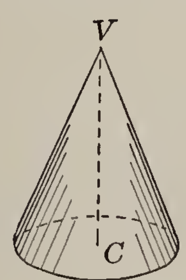


FIG. 115

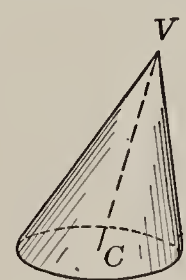


FIG. 116

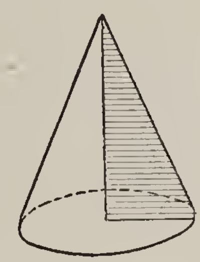


FIG. 117

cone is said to be **oblique,** Fig. 116. A right circular cone may be generated by revolving a right triangle about one of its sides as an axis, Fig. 117. Hence it is also called a **cone of revolution.** The distance from the vertex to a point of the base of a cone of revolution is called the **slant height.**

EXERCISES

1. The altitude of a right circular cone is 10 in. and the radius of the base is 4 inches. Find the slant height.

2. The slant height of a right circular cone is 50 in. and is twice as long as the radius of the base. Find the altitude.

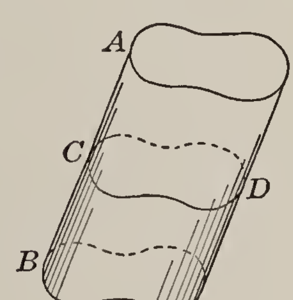


FIG. 118

251. Cylinder. If a straight line, AB, Fig. 118, so moves as to remain parallel to a fixed straight line and to touch a curve, CD, whose plane does not pass through the

fixed line, the moving straight line generates a *cylindrical surface.* The moving line, AB, is the *generatrix.* The generatrix in any one of its positions is an *element* of the surface. A solid bounded by a cylindrical surface and two parallel planes is a **cylinder,** Fig. 119. The cylindrical surface is the *lateral surface* of the cylinder, the two plane curves, AB and $A'B'$, are the *bases.*

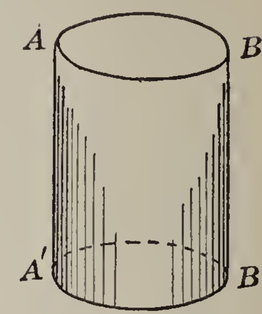


FIG. 119

252. Classification of cylinders. If the base is a circle and if the elements are perpendicular to the plane of the circle, the cylinder is called a **right circular cylinder,** Fig. 120. A cylinder whose elements are oblique to the base is an **oblique cylinder,** Fig. 121. A right circular cylinder may be generated by revolving a rectangle, as $acc'a'$, Fig. 120, about one of the sides. Hence it is also called a **cylinder of revolution.**

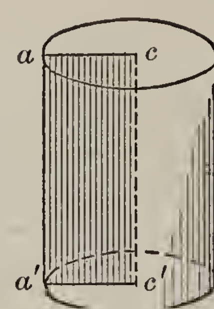


FIG. 120

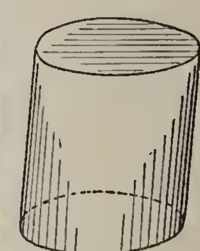


FIG. 121

Give examples of objects of cylindrical form.

253. Prism. *A prismatic surface,* Fig. 122, may be generated by moving a line parallel to itself, so that it always touches a given convex polygon, as $ABCDE$. A solid bounded by parallel planes and a prismatic surface is a **prism,** Fig. 123. The terms *base,*

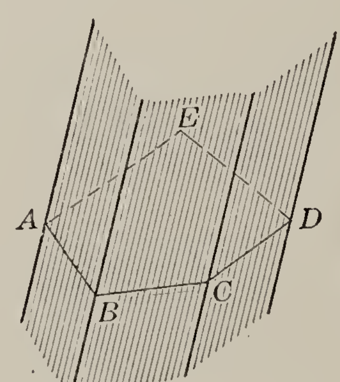


FIG. 122

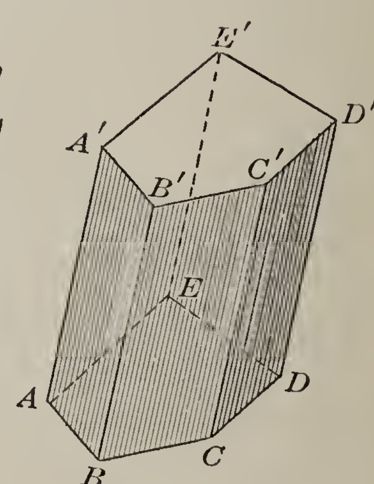


FIG. 123

lateral surface, and *altitude* have the same meaning for the prism as for the cylinder.

254. Classification of prisms. A prism is **right,** Fig. 124, or **oblique,** Fig. 125, according as the lateral edges are perpendicular or oblique to the planes of the bases. A prism is said to be **triangular, quadrangular,** etc., according as the base is a triangle, quadrilateral, etc.

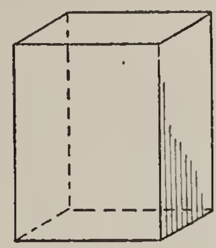


FIG. 124

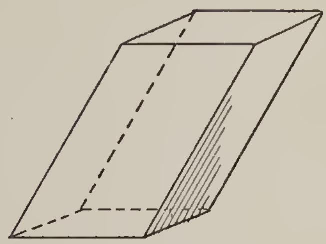


FIG. 125

EXERCISES

1. Prove that *the lateral edges of a prism are equal.*

2. Prove that *the lateral edges of a right prism are equal to the altitude.*

3. Prove that *the lateral faces of a prism are parallelograms.*

4. What is the locus of a straight line parallel to a given straight line and at a given distance from it?

5. What is the locus of a point having a given distance from a given straight line?

6. Prove that a lateral edge of a prism is parallel to every lateral face not containing it.

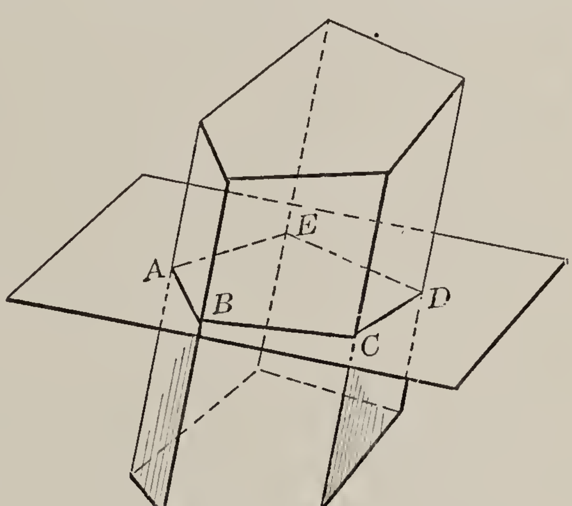


FIG. 126

Sections Made by a Plane

255. Section. The intersection of a plane with the surface of a solid is called a **section,** as $ABCDE$, Fig. 126.

256. Right section. If the plane of a section is at right angles to the lateral edges of a prism or to the elements of a cylinder, it is a **right section.**

257. Theorem: *The sections of a prism made by parallel planes cutting all the lateral edges are congruent.*

Given the prism, AB, Fig. 127, plane $CF \parallel$ plane $C'F'$; sections $CDEFG$ and $C'D'E'F'G'$.

To prove that

$CDEFG \cong C'D'E'F'G'$.

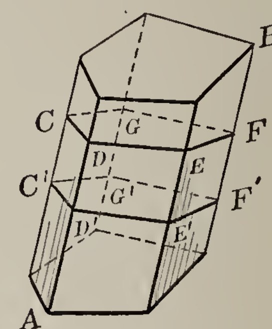


Fig. 127

Proof: Prove

$CD \parallel C'D'$, $DE \parallel D'E'$, etc.

Prove

$CD = C'D'$, $DE = D'E'$, etc.

Prove

$\angle CDE = \angle C'D'E'$, $\angle DEF = \angle D'E'F'$, etc. (§ 545).

Hence

$CDEFG \cong C'D'E'F'G'$. Why?

EXERCISES

Prove the following:

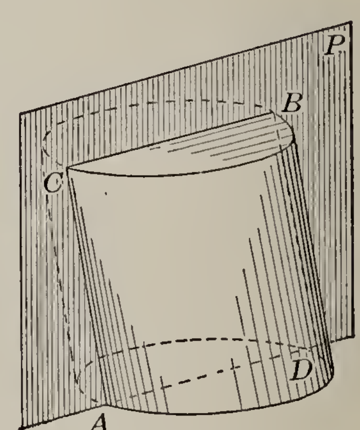


Fig. 128

1. *The right sections of a prism are congruent.*

2. *A section of a prism parallel to the base is congruent to the base.*

3. *The bases of a prism are congruent.*

4. A section of a prism made by a plane parallel to a lateral edge is a parallelogram.

258. Theorem: *A section of a cylinder made by a plane passing through an element is a parallelogram.*

Given the cylinder AB, Fig. 128; the element DB; and plane P passing through DB.

To prove that the intersection of P with the surface of the cylinder AB is a parallelogram.

Proof: Plane P intersects the base in the straight line AD. Why?

Through A draw line AC parallel to DB.

AC must lie in plane P. Why?

AC is also an element. Why?

Therefore AC lies in the cylindrical surface. Why?

Hence AC must be the intersection of P with the cylindrical surface. Why?

Draw CB and show that it is the intersection of P with the upper base of the cylinder.

Show that $ADBC$ is a parallelogram.

EXERCISE

Show that $ADBC$, Fig. 128, is a rectangle if AB is a right cylinder.

259. Theorem: *Sections of a cylinder made by parallel planes cutting all elements are congruent.*

Given the cylinder AB, Fig. 129; section $CD \parallel$ section $C'D'$.

To prove that $CD \cong C'D'$.

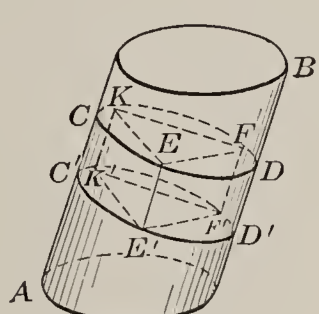


Fig. 129

Proof: Let E' and F' be two fixed points on $C'D'$ and let K' be any other point on $C'D'$.

Draw the elements through E', F', and K', meeting CD in E, F, and K respectively.

Draw triangles EFK and $E'F'K'$.

Prove $\triangle EFK \cong \triangle E'F'K'$ (*s.s.s.*).

Imagine CD placed on $C'D'$ with E coinciding with E' and F with F'.

Then K must fall on K'. Why?

Hence every point on CD can be made to coincide with the corresponding point on $C'D'$, and $CD \cong C'D'$.

EXERCISES

Show that the following statements are special cases of the theorem just proved:

1. *Sections of a cylinder parallel to the base are congruent to the base.*

2. *The bases of a cylinder are congruent.*

3. *All right sections of a cylinder are congruent.*

260. Theorem: *If a pyramid is cut by a plane parallel to the base:*

1. *The edges and altitude are divided proportionally.*

2. *The section is a polygon similar to the base.*

3. *The areas of the section and the base are proportional to the squares of the distances from the vertex.*

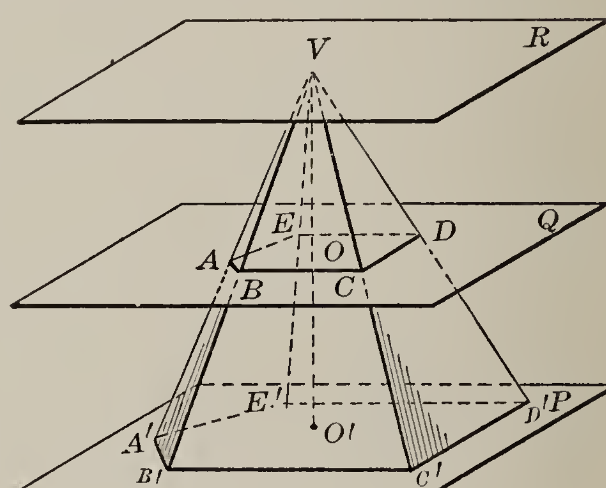


Fig. 130

Given the pyramid $V\text{-}A'B'C'D'E'$, Fig. 130, and plane $Q \parallel$ plane P; $VO' \perp P$.

1. **To prove** that $\dfrac{VA}{VA'} = \dfrac{VB}{VB'} = \dfrac{VC}{VC'} = \ldots\ldots = \dfrac{VO}{VO'}$.

Proof: Through V draw plane $R \parallel$ planes P and Q. Show that the conclusion follows (see § 524).

2. **To prove** that $ABCDE \backsim A'B'C'D'E'$.

Proof: Show that $AB \parallel A'B'$; $BC \parallel B'C'$; etc.

$\therefore \triangle VAB \backsim \triangle VA'B'$; $\triangle VBC \backsim \triangle VB'C'$, etc.

$$\therefore \quad \frac{VB}{VB'} = \frac{AB}{A'B'}; \quad \frac{VB}{VB'} = \frac{BC}{B'C'} \qquad \text{Why?}$$

$$\therefore \quad \frac{AB}{A'B'} = \frac{BC}{B'C'} \qquad \text{Why?}$$

Similarly show that

$$\frac{BC}{B'C'}=\frac{CD}{C'D'}, \text{ etc.}$$

Prove that $\angle ABC=\angle A'B'C'$; $\angle BCD=\angle B'C'D'$; etc.

$$\therefore\ ABCDE \backsim A'B'C'D'E'. \qquad \text{Why?}$$

3. **To prove** that $\dfrac{ABCDE}{A'B'C'D'E'}=\dfrac{\overline{VO}^2}{\overline{VO'}^2}$.

Proof:

$$\frac{ABCDE}{A'B'C'D'E'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad \text{Why?}$$

$$\frac{\overline{AB}^2}{\overline{A'B'}^2}=\frac{\overline{VB}^2}{\overline{VB'}^2}=\frac{\overline{VO}^2}{\overline{VO'}^2}. \qquad \text{Why?}$$

$$\therefore\ \frac{ABCDE}{A'B'C'D'E'}=\frac{\overline{VO}^2}{\overline{VO'}^2}. \qquad \text{Why?}$$

261. Theorem: *A section of a cone made by a plane passing through the vertex is a triangle.*

Given the cone $V\text{-}AB$ cut by plane P, Fig. 131.

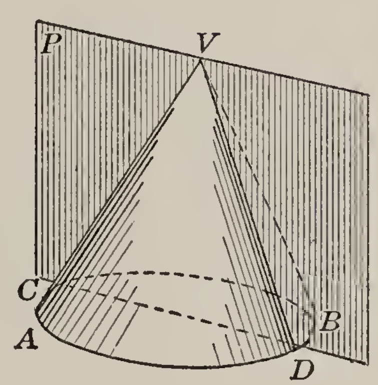


FIG. 131

To prove that the section is a triangle.

Proof: Plane P intersects the base ACB in the straight line, CD.

Draw the straight lines VC and VD.

Then VC and VD are elements. Why?

Therefore VC and VD lie in the conical surface. Why?

But they also lie in plane P. Why?

Hence the straight lines VC and VD are the intersections of P with the conical surface, and the section is a triangle. Why?

262. Theorem: *A section of a circular cone made by a plane parallel to the base is a circle.*

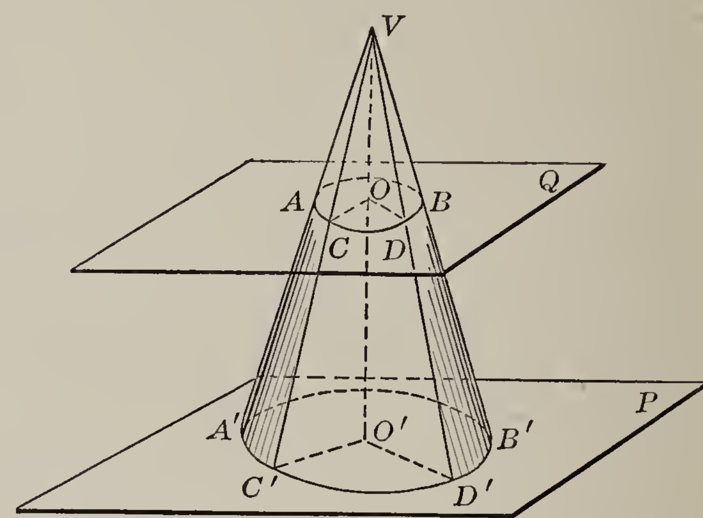


Fig. 132

Given the circular cone $V\text{-}A'B'$, Fig. 132, and plane $Q \parallel$ plane P.

To prove that the section $ACDB$ is a circle.

Proof: Take any two points, C and D, on AB.

Draw the elements VCC' and VDD'.

Draw VO' intersecting the plane of AB in O.

Draw CO, DO, $C'O'$, $D'O'$.

Prove that $\triangle VOC \sim \triangle VO'C'$; $\triangle VOD \sim \triangle VO'D'$.

Then $\dfrac{VO}{VO'} = \dfrac{CO}{C'O'}$. Why?

$\dfrac{VO}{VO'} = \dfrac{DO}{D'O'}$. Why?

$\therefore \dfrac{CO}{C'O'} = \dfrac{DO}{D'O'}$. Why?

Since $C'O' = D'O'$, it follows that $CO = DO$.

$\therefore$ the section $ACDB$ is a circle. Why?

EXERCISES

1. The area of the base of a pyramid is 110 square feet. The area of the section of the pyramid parallel to the base and 5 ft. from it is 80 square feet. Find the altitude to two decimal places.

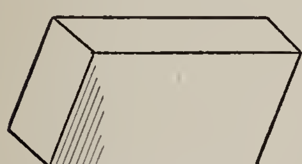
2. The base of a pyramid is 50 sq. in. and the altitude 6 inches. How far from the vertex must a plane be passed that the area of the section may be half as large as the area of the base?

EXERCISES

1. Show that *the axis of a right circular cone passes through the center of every section parallel to the base.*

2. Prove that the radius of the section of a circular cone made by a plane parallel to the base, and the radius of the base are proportional to the distances from the vertex to the cutting plane and to the plane of the base.

263. Parallelopipeds. A prism whose bases are parallelograms is a **parallelopiped,** Figs. 133 to 136. A parallelopiped whose lateral edges are perpendicular to the bases is a **right parallelopiped,** Fig. 134. A right parallelopiped whose bases are rectangles is a **rectangular parallelopiped,** Fig. 135. A parallelopiped all of whose faces are squares is a **cube,** Fig. 136.

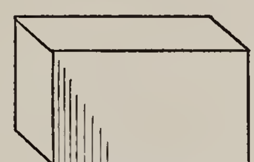


Parallelopiped

FIG. 133

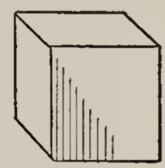
Right Parallelopiped

FIG. 134



Rectangular Parallelopiped

FIG. 135



Cube

FIG. 136

EXERCISES

1. State the difference between a right parallelopiped and a rectangular parallelopiped.

2. Show that the faces of a rectangular parallelopiped are *all* rectangles.

3. Prove that *the opposite faces of a parallelopiped are parallel.*

Use § 545.

4. Prove that *a section of a parallelopiped made by a plane cutting four parallel edges is a parallelogram.*

5. Prove that the diagonals of a cube are equal.

6. Find the diagonal of a cube whose edge is 2; 3.4; e.

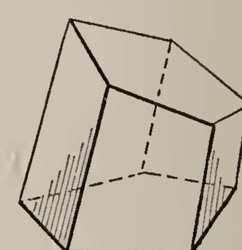
7. Prove that the square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of three edges meeting in the same vertex.

8. Find the diagonal of a rectangular parallelopiped whose edges are 6, 8, and 10 respectively.

9. Prove that the diagonals of a rectangular parallelopiped are equal and bisect each other.

10. Find the length of the diagonal of a rectangular parallelopiped whose edges from any vertex are 4, 6, and 8.

11. Find the edge of a cube whose diagonal is 12 inches.



Truncated Prism

FIG. 137

264. Truncated prism. A portion of a *prism* included between the plane of the base and the plane of a section *not parallel to the base* is a **truncated prism,** Fig. 137.

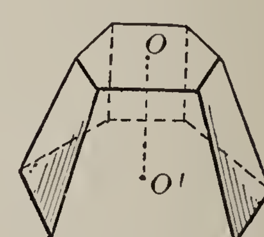


FIG. 138

265. Frustum of a pyramid. Altitude. The portion of a *pyramid* included between the plane of the base and the plane of a section *parallel to the base* is a **frustum of a pyramid**, Fig. 138. The perpendicular, OO', intercepted between the planes of the bases is called the **altitude** of the frustum.

EXERCISES

1. Show that the lateral faces of a frustum of a pyramid are trapezoids.

2. Show that the lateral faces of a frustum of a *regular* pyramid are *congruent* trapezoids.

3. Prove that a plane bisecting the altitude and parallel to the plane of the bases of a frustum of a pyramid forms a section whose perimeter is half the sum of the perimeters of the bases.

266. Slant height of a frustum. The altitude, AB, Fig. 139, of a lateral face of a frustum of a regular pyramid is the **slant height** of the frustum.

267. Frustum of a cone. The portion of a cone included between the plane of the base and the plane of a section parallel to the base is a **frustum of a cone,** Fig. 140.

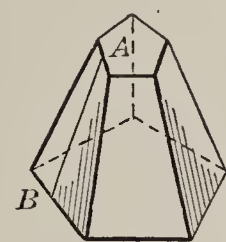


FIG. 139

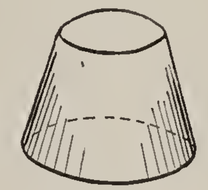


FIG. 140

268. Sections of a cone. In the discussion of the plane sections of a right circular cone the following cases may be considered:

Let P be a plane perpendicular to plane AVB, Fig. 141.

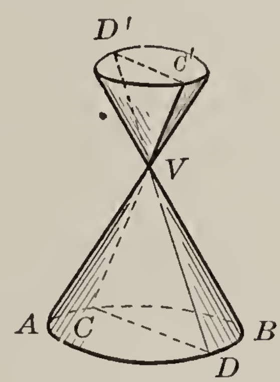


FIG. 141

1. If P, Fig. 141, passes through the vertex V and an element VD, it cuts the surface in *two intersecting straight lines*, as DD' and CC'.

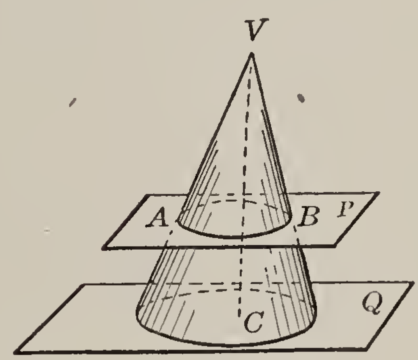


FIG. 142

2. If P does not pass through the vertex V, and if it is perpendicular to the axis VC, the section is a *circle*, Fig. 142.

3. If P, Fig. 143, is not perpendicular to the axis, but meets both of the elements VA and VB, the section is an *ellipse*.

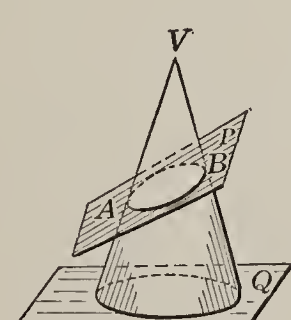


FIG. 143

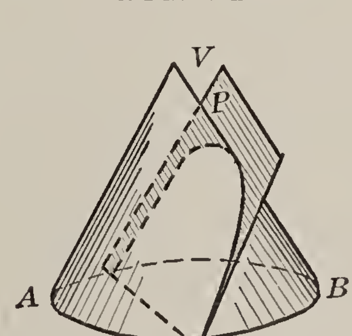


FIG. 144

4. If P, Fig. 144, is parallel to one of the elements, the section is a *parabola*.

5. If plane P, Fig. 145, meets some of the elements produced, the section is a *hyperbola.*

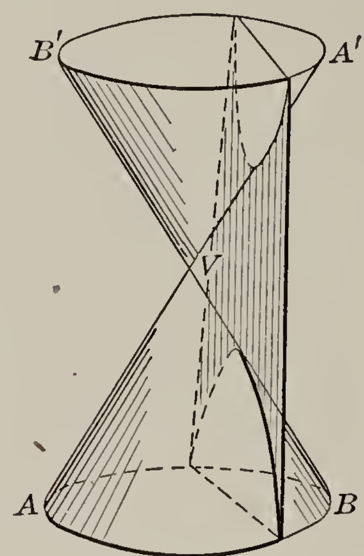


Fig. 145

Thus we have the following sections of a cone:

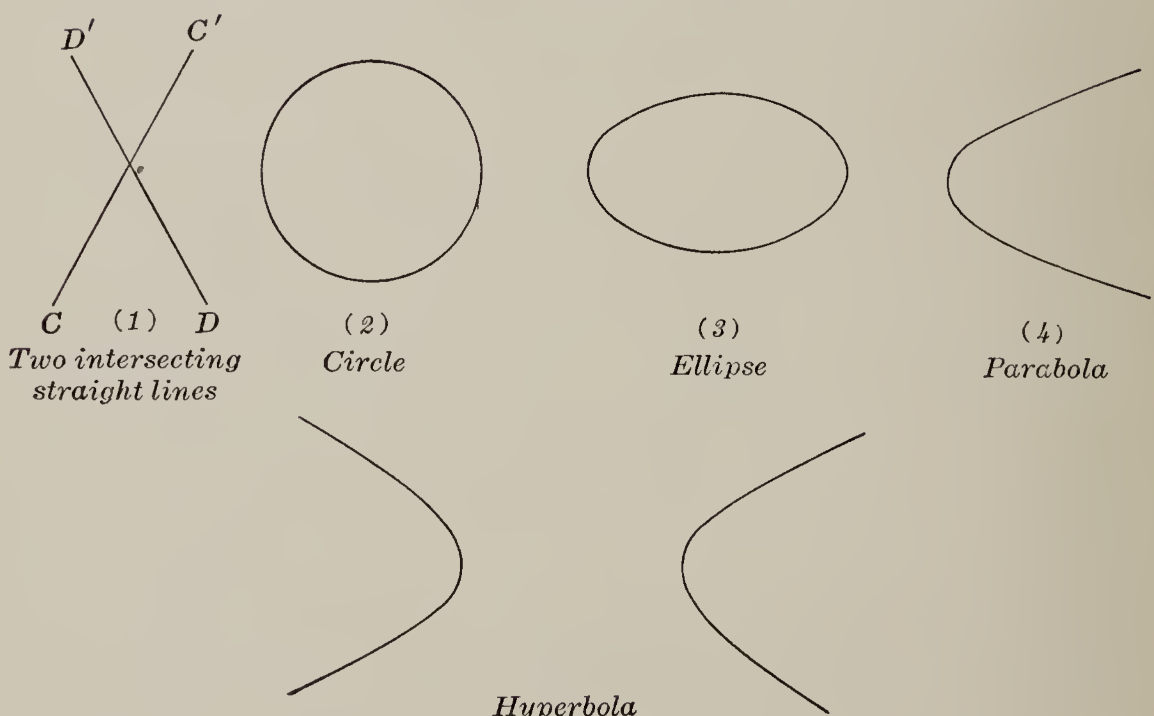


Fig. 146

It was seen in chapter XI that these curves represent graphically quadratic equations in two unknowns.

A very extensive study of these sections is made in *analytic geometry.*

Areas

269. Theorem: *The lateral area of a prism is equal to the perimeter of a right section multiplied by the lateral edge.*

In symbols this may be expressed by the equation

$$L = p \cdot e,$$

L denoting the lateral area, p the perimeter of the right section, and e the length of a lateral edge.

Given the prism AD'; the right section FK, Fig. 147.

To prove that $L = p \cdot e$.

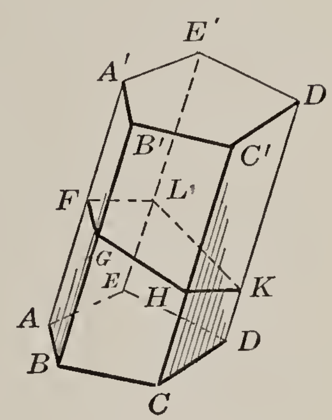


Fig. 147

Proof: Show that the lateral edges are equal.

Show that the lateral faces are parallelograms.

Show that the sides of the section FK are the altitudes of these parallelograms.

Hence $\quad AB' = \overline{FG} \cdot \overline{BB'} = \overline{FG} \cdot e,$

$\quad BC' = \overline{GH} \cdot \overline{CC'} = \overline{GH} \cdot e,$ etc.

Adding, $AB' + BC' +$ etc. $= (FG + GH + \text{etc.})e,$

or $$L = p \cdot e.$$

EXERCISES

1. Prove that *the lateral area of a right prism is equal to the perimeter of the base by the altitude.*

2. Find the lateral area of a prism whose lateral edge is 18 cm. and whose right section has a perimeter equal to 29 centimeters.

3. Find L (1) if $e = 2.75$, $p = 5.26$

(2) if $e = 5\frac{1}{4}$, $p = 10\frac{3}{7}$

(3) if $e = 12.14$, $p = 25\frac{19}{100}$

4. Find p (1) if $L=20.26$, $e=12.48$

(2) if $L=19\frac{4}{7}$, $e=6.92$

5. Find the lateral area of a column having the form of a hexagonal right prism, if one side of the base is 5 in., and if the altitude is 8 feet.

6. Find the total surface of a cube whose edge is 2.6 centimeters.

7. Find the total area of a right triangular prism, if the base is an equilateral triangle whose side $a=2.7$ in. and whose altitude $h=8.4$ inches.

8. How many square inches of copper lining will be required to line the sides and base of a tank 9 in. high, $9\frac{3}{4}$ in. wide, and 20 in. long?

9. How many square feet of lead will be required to line a rectangular cistern $9\frac{1}{2}$ ft. long, 7 ft. wide, and 5 ft. deep?

270. Theorem: *The lateral area of a regular pyramid is equal to one-half the product of the slant height by the perimeter of the base.*

In symbols,

$$L=\frac{1}{2}s\cdot p,$$

L denoting the lateral area, s the slant height, and p the perimeter of the base.

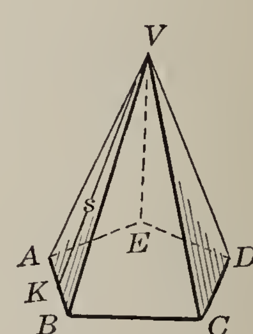


Fig. 148

Given the regular pyramid

$V\text{-}ABCDE,$

Fig. 148; the slant height VK.

To prove that $L=\frac{1}{2}s\cdot p.$

Proof: Show that $L=\frac{1}{2}s\cdot AB+\frac{1}{2}s\cdot BC+\frac{1}{2}s\cdot CD$ + etc.

$$\therefore\ L=\frac{1}{2}s\cdot p$$

271. Theorem: *The lateral area of the frustum of a regular pyramid is equal to one-half the product of the sum of the perimeters of the bases by the slant height,* or in symbols,

$$L = \frac{1}{2}(p_1 + p_2)s.$$

Prove.

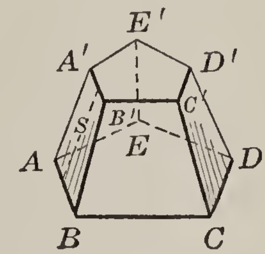


FIG. 149

EXERCISES

1. The slant height of a regular triangular pyramid is 8 feet. The side of the base is 3 feet. Find the lateral area.

2. The altitude of a regular pyramid is 5 feet. The base is a regular hexagon whose side is 6 feet. Find the lateral area.

3. The altitude of a regular pyramid is 8 feet. The base is a square whose area is 25 square feet. Find the lateral area.

4. The base of a regular pyramid is a square whose side is 6. The slant height makes an angle of 45° with the plane of the base. Find the lateral area.

5. The base of a regular pyramid is a square whose area is 900. The altitude is 12. Find the lateral area.

6. The sides of the bases of a frustum of a regular hexagonal pyramid are 6 and 14 respectively. The slant height is 20. Find the lateral area and total area.

7. Find the cost of painting a church spire at the rate of 20 cents per square yard. The altitude of the spire is 80 ft. and a side of its hexagonal base is 10 feet.

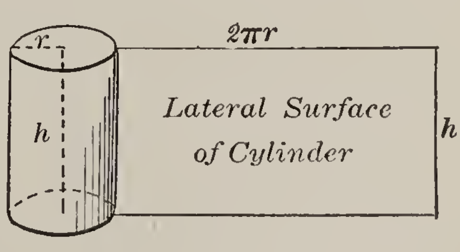


FIG. 150

272. Lateral area of a right cylinder and of a right cone. The lateral area of a right cylinder and of a right cone may be found by rolling the lateral surface along a plane. The **lateral surface of a right cylinder** is found to be a rectangle, Fig. 150, whose width

is equal to the altitude of the cylinder and whose length is equal to the length of the circle forming the base of the cylinder.

Hence, $$L = 2\pi rh,$$

where L is the lateral area, r the radius of the base, and h the altitude of the cylinder.

Similarly **the lateral surface of a right cone** is found to be a sector of a circle, Fig. 151, whose arc equals the length of the circle forming the base of the cone, and whose radius is equal to the slant height of the cone.

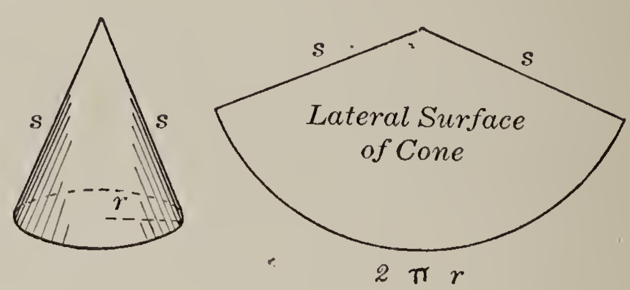


Fig. 151

Hence $L = \pi rs$.

EXERCISES

1. Roll an oblique cylinder along a plane and make a drawing of the lateral surface.

2. Make a drawing of the lateral surface of an oblique cone.

3. The extreme length of a clothes boiler is 24 in., the width is $11\frac{1}{2}$ in., and the depth $12\frac{3}{4}$ inches. The ends of the boiler are semicircular. Allowance has to be made for locking as follows: $1\frac{1}{2}$ inches on the width of the side piece and 1 in. on the length; $\frac{1}{2}$ in. all around the bottom piece. How much tin is required to make the boiler?

273. Lateral area of a frustum of a right cone. Show that the lateral surface of a frustum of a right cone, Fig. 152, is the difference of the lateral surfaces of two right cones.

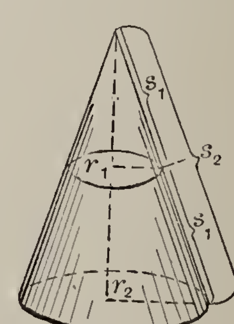


Fig. 152

Hence the lateral area is given by the formula

$$L=\pi(s_2r_2-s_1r_1).$$

Since s_1 and s_2 are not parts of the frustum, this formula will be changed to a different form, as follows:

$$\frac{s_1}{r_1}=\frac{s_2}{r_2} \qquad \text{Why?}$$

$$\therefore\ s_2r_1-s_1r_2=0 \qquad \text{Why?}$$

$$\begin{aligned}\therefore\ L&=\pi(s_2r_2+s_2r_1-s_1r_2-s_1r_1)\\ &=\pi[s_2(r_2+r_1)-s_1(r_2+r_1)]\\ &=\pi(r_2+r_1)(s_2-s_1)\\ \therefore\ \boldsymbol{L}&=\boldsymbol{\pi(r_1+r_2)s}\end{aligned}$$

This may be written:

$$\boldsymbol{L=\tfrac{1}{2}(2\pi r_1+2\pi r_2)s.}$$

Hence *the lateral area of a frustrum of a right circular cone is equal to one-half the product of the slant height and the sum of the perimeters of the bases.*

EXERCISES

1. Show that the total area of a cylinder of revolution is given by the formula $T=2\pi r(h+r)$, h being the altitude and r the radius of the base.

2. Show that the total area of a cone of revolution is given by the formula $T=\pi r(s+r)$, s being the slant height and r the radius of the base. State in words the law expressed by this formula.

3. Show that the lateral area of a frustum of a cone of revolution is equal to the slant height multiplied by the length of a

circle obtained by cutting the frustum by a plane at equal distances from the bases.

Show that $r=\frac{1}{2}(r_1+r_2)$, Fig. 153.

Hence $r_1+r_2=2r$.

Substituting this in the equation $L=\pi(r_1+r_2)s$, it follows that

$$L=2\pi r \cdot s.$$

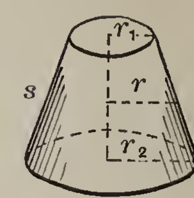


FIG. 153

4. How much metal is required to construct a galvanized iron pail which is 9 in. in diameter at the top, $8\frac{1}{2}$ in. at the bottom, and 11 in. in the slant height, allowing $1\frac{1}{2}$ in. on the width, 1 in. on the length of the side piece for locking, and 1 in. on the diameter of the bottom piece?

5. The lateral area of a frustum of a right circular cone is 60π square feet. If the radii of the bases are 4 ft. and 6 ft. respectively, find the slant height.

274. Similar cylinders. Two right circular cylinders are **similar** if they are generated by revolving two similar rectangles about corresponding sides, Fig. 154.

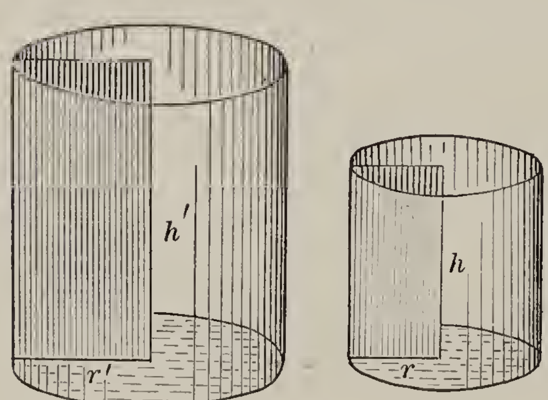


FIG. 154

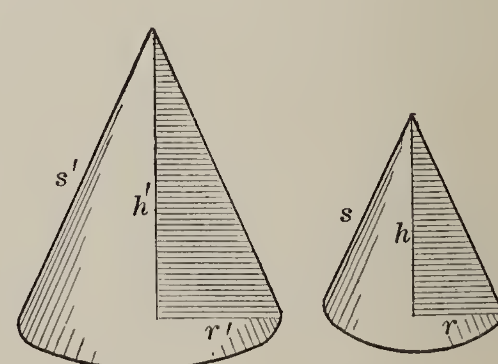


FIG. 155

275. Similar cones. Two right circular cones are **similar** if they are generated by revolving two similar right triangles about corresponding sides, Fig. 155.

276. Theorem: *The lateral areas, or the total areas, of similar right circular cylinders, or cones, are proportional to the squares of the altitudes, or to the squares of the radii of the bases.*

Proof: Denoting the radii by r and r', Fig. 154, the altitude by h and h', the lateral areas by L and L', and the total areas by T and T', show that

$$\frac{L}{L'}=\frac{2\pi rh}{2\pi r'h'}=\frac{rh}{r'h'}=\frac{r}{r'}\times\frac{h}{h'}=\frac{r^2}{r'^2}=\frac{h^2}{h'^2},$$

$$\frac{T}{T'}=\frac{2\pi r(h+r)}{2\pi r'(h'+r')}=\frac{r(h+r)}{r'(h'+r')}=\frac{r}{r'}\times\frac{h+r}{h'+r'}.$$

Since $\frac{h}{h'}=\frac{r}{r'}$, it follows that $\frac{h+r}{h'+r'}=\frac{r}{r'}=\frac{h}{h'}$.

By substitution,

$$\frac{T}{T'}=\frac{r^2}{r'^2}=\frac{h^2}{h'^2}.$$

The proof for the cones, Fig. 155, is similar and is left to the student.

EXERCISES

1. Show that the lateral areas, or total areas, of two similar right cones are to each other as the squares of the slant heights.

2. How many square feet of surface are there in a tank formed by a cylinder and cone of the dimensions and shape shown in Fig. 156?

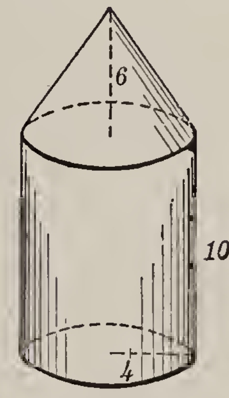


FIG. 156

3. How many square feet of material, not allowing for waste, have been used in the construction of a silo, Fig. 157, the diameter of whose base is 16 ft., whose total height is 24 ft., and the height of whose roof is 8 feet?

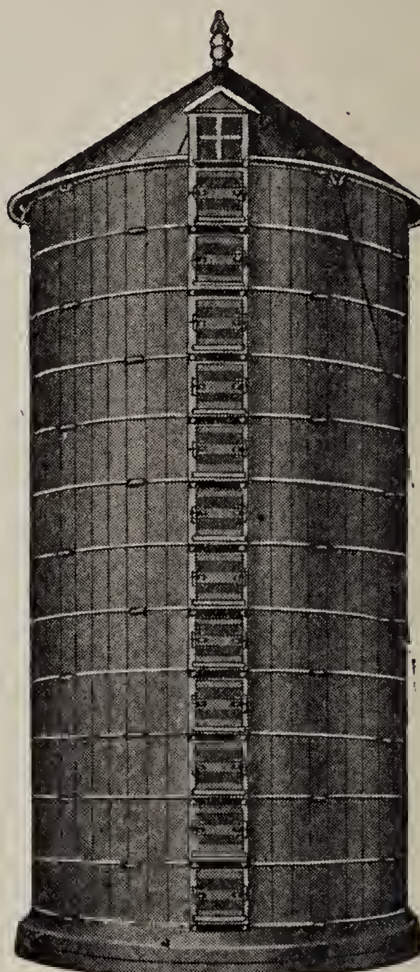


FIG. 157

GENERAL EXERCISES

‡277. Solve the following problems:

1. Find the lateral surface and the total surface of a cylinder of revolution if $h=8.5$ in. and $r=5.3$ inches.

2. Find the lateral surface of a quadrangular right pyramid, the side of whose base is 7 cm. and whose altitude is 6.8 centimeters.

3. Find the lateral surface and the total surface of a right cone whose radius is 4.2 and whose altitude is 5.7.

4. Find the lateral surface of a frustum of a pyramid whose altitude is 10 and whose bases are squares with sides equal to 4 and 6 respectively.

5. The altitude of a prism is 40. The base is a right triangle having the sides of the right angle equal to 36 and 43 respectively. Find the lateral and total area.

6. The base of a right prism is a regular hexagon whose side is 6. The altitude of the prism is 10. Find the lateral and total area.

7. The curved surface of a cylindrical column made of granite is to be polished. What will be the expense at the rate of 60 cents per square foot, if the diameter of the base is 4.5 ft. and the column is 24 ft. high?

8. The great pyramid of Cheops is about 460 ft. high. The base is a square whose side is 746 ft. long. What is the lateral area?

9. A steeple is of the form of a regular hexagonal pyramid. The perimeter of the base is 60 feet. The slant height is 48 feet. How many square feet must be allowed for slating the steeple?

10. At 26 cents a square yard what will be the cost of painting a gas tank of the form of a right circular cylinder, if the height is 72 ft. and the diameter of the base 45 feet?

11. A funnel is 8 in. in diameter at the widest end, $1\frac{1}{2}$ in. at the spout, and 1 in. at the smaller end of the spout. The slant height of the funnel is 6 in. and that of the spout is 4 inches. Allowing for locking $\frac{1}{2}$ in. on the length and width of each part, find the amount of tin needed to make the funnel.

Fig. 158

12. How far from the vertex of a right circular cone must a plane be passed parallel to the base and so that the lateral area of the small cone cut off shall be equivalent to the lateral area plus one base of a right circular cylinder? The altitude of the cone is 12 in., the radius of the base 8 inches. The altitude of the cylinder is 4 in. and the radius of its base is 2 inches. How far from the vertex of the cone must the plane be passed?

13. A windmill water-supply tank, Fig. 158, is 8 ft. in diameter and 12 ft. high. The roof is 9 ft. in diameter and 3 ft. high. How much material was used in its construction?

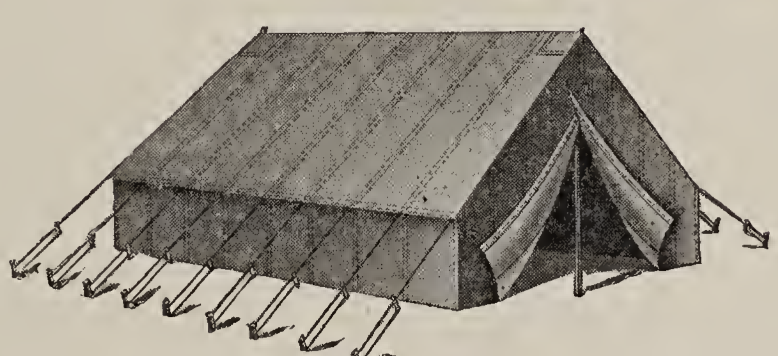


Fig. 159

14. The width and length of a tent, Fig. 159, are 12 ft. and 18 ft. respectively. The height of the pole is 8 ft. and the height

of the wall $3\frac{1}{2}$ feet. How much material was used in making the tent?

Surfaces of Revolution

278. Surface of revolution. If a line segment, AB Figs. 160–164, revolves about a straight line, CD, in the same plane, as an axis, every point of the segment describes a circle whose plane is perpendicular to the axis. Why?

The *surface* generated by the segment is a **surface of revolution.** According to the position of the segment with reference to the axis, the surface of revolution of the segment is a *lateral surface of a right cone,* Fig. 160, a

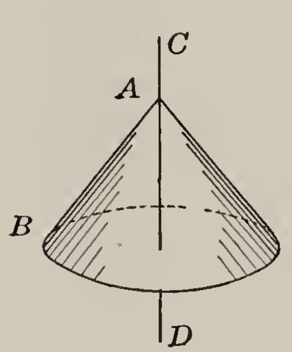


FIG. 160

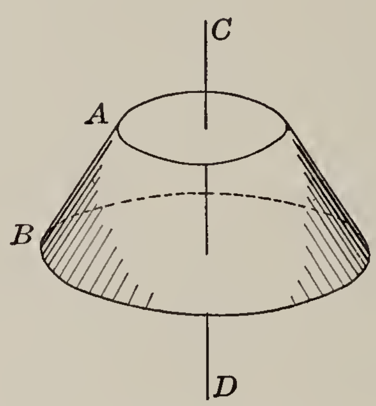


FIG. 161

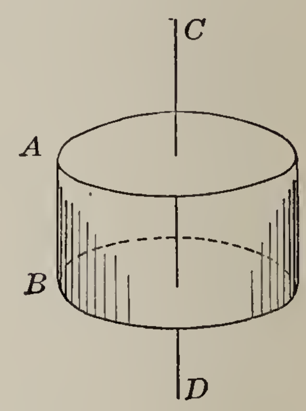


FIG. 162

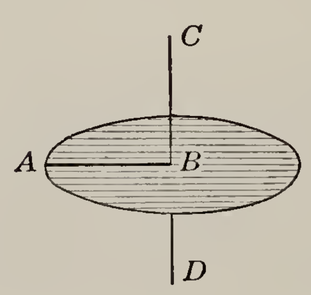


FIG. 163

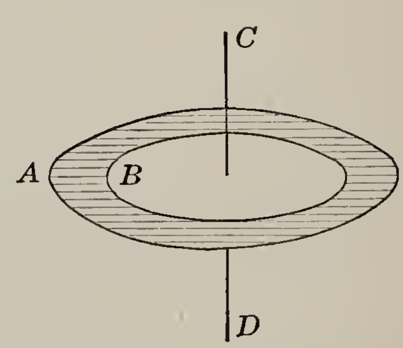


FIG. 164

frustum of a right cone, Fig. 161, *a right cylinder,* Fig. 162, a *surface of a circle,* Fig. 163, or a *circular ring,* Fig. 164.

279. Theorem: *If half of a regular polygon having an even number of sides is revolved about a diagonal joining two opposite vertices, the area of the surface thus generated is equal to the product of the diagonal by the length of the circle inscribed in the polygon.*

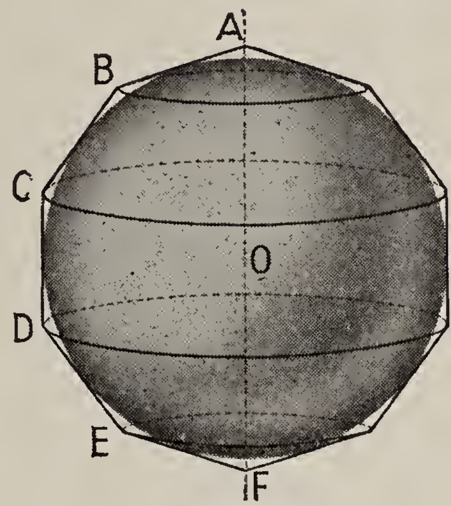


FIG. 165

Proof: The surface, Fig. 165, is composed of cones, frustum of cones, and cylinders. Hence the area may be found by adding the areas of these cones, frustums, and cylinders.

It will be shown that *one formula* may be used to find the lateral surface of each.

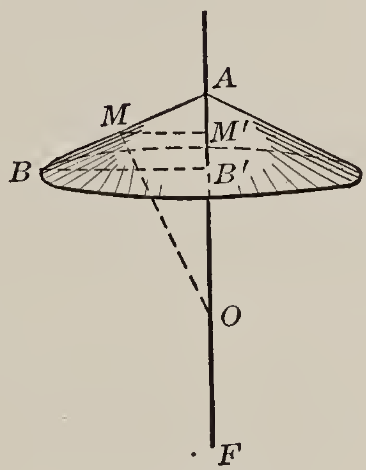


FIG. 166

1. The area of the surface generated by AB, Fig. 166, is given by

$$L_1 = \pi \overline{BB'} \times \overline{AB}, \text{ § 272.}$$

Bisect AB at M.

Draw $MM' \perp AF$.

Show that $MM' = \frac{1}{2}\overline{BB'}$.

Then $L_1 = 2\pi \overline{MM'} \times \overline{AB}$. Why?

Draw $MO \perp AB$.

$\triangle OMM' \backsim \triangle ABB'$. Why?

$\therefore \dfrac{AB}{MO} = \dfrac{AB'}{MM'}$. Why?

$\therefore \overline{MM'} \times \overline{AB} = \overline{MO} \times \overline{AB'}$. Why?

$\therefore L_1 = 2\pi \overline{MO} \times \overline{AB'}$. Why?

2. The area of the surface generated by BC, Fig. 167, is given by

$$L = \pi(\overline{CC'} + \overline{BB'})\overline{BC}, \text{ § 273.}$$

Bisect BC at M. Draw $MM' \perp AF$.

Show that $CC' + BB' = 2MM'$.

Then $L_2 = 2\pi\overline{MM'} \times \overline{BC}$.

Draw $BB'' \perp CC'$.

Then $\triangle CBB'' \backsim \triangle OMM'$. Why?

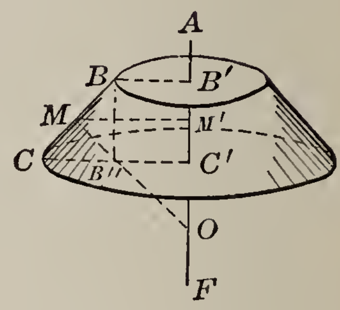


Fig. 167

$$\therefore \frac{CB}{MO} = \frac{BB''}{MM'} = \frac{B'C'}{MM'}.$$

$$\therefore \overline{MM'} \times \overline{CB} = \overline{MO} \times \overline{B'C'}.$$

$$\therefore L_2 = 2\pi\overline{MO} \times \overline{B'C'}$$

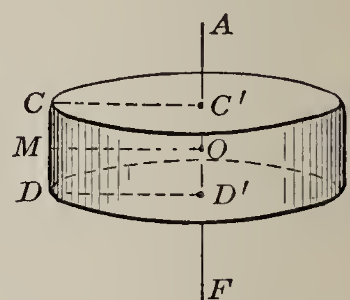


Fig. 168

3. The area of the surface generated by CD, Fig. 168, is given by

$$L_3 = 2\pi\overline{DD'} \times \overline{CD}.$$

Bisect CD at M. Draw MO.

Then $L_3 = 2\pi\overline{MO} \times \overline{CD}$.

Similarly the area of the remaining part of the surface is found.

Thus,
$$L_1 = 2\pi\overline{MO} \times \overline{AB'}$$
$$L_2 = 2\pi\overline{MO} \times \overline{B'C'}$$
$$L_3 = 2\pi\overline{MO} \times \overline{C'D'}, \text{ etc.}$$

Adding, $L = 2\pi\overline{MO}(AB' + B'C' + \ldots . E'F)$,

Fig. 165

or $\mathbf{L = 2\pi\overline{MO} \times \overline{AF}}$

280. Area of the surface of a sphere. To find the area of the surface of a sphere inscribe in a semicircle half a regular polygon as $ABCDE$, Fig. 169.

The area of the surface generated by the polygon $ABCDE$ is $2\pi\overline{MO}\times\overline{AE}$.

Let the number of sides of the polygon be increased indefinitely.

Then the polygon approaches the circle as a limit.

The area of the surface generated by the polygon approaches as a limit the area of the surface of the sphere generated by the semicircle.

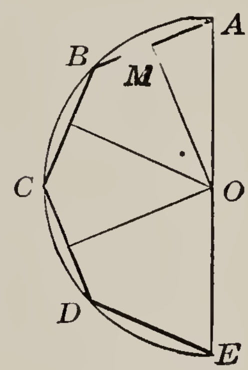


Fig. 169

MO approaches the radius r as a limit.

Hence $2\pi\overline{MO}$ approaches $2\pi r$, and $2\pi\overline{MO}\times\overline{AE}$ approaches $2\pi r\times\overline{AE}$, which is equal to $2\pi r\times 2r=4\pi r^2$. Hence the area of the surface generated by $ABCDE$ approaches $4\pi r^2$ as a limit.

Thus the preceding discussion leads to the following theorem:

The area of the surface of a sphere is equal to the product of the diameter by the length of a great circle, or

$$S=4\pi r^2.$$

281. Zone. A portion of a spherical surface included between two parallel planes is a **zone,** Fig. 170. The distance between the planes is the **altitude** of the zone. The sections made by the planes are the **bases** of the zone. If one of the planes is tangent to the spherical surface, the zone is said to have *one base.*

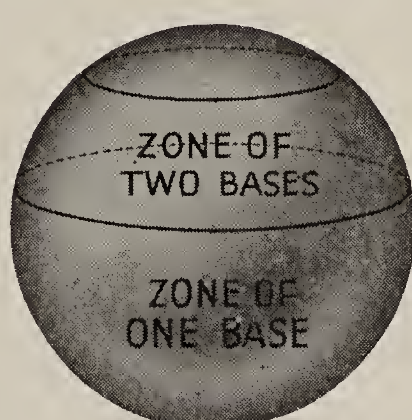


Fig. 170

EXERCISES

1. Show that *the area of a zone is equal to the product of the altitude by the length of a great circle, or* $Z = 2\pi rh$. Determine the area of a zone whose altitude is 12, if the radius of the sphere is 15.

2. *The areas of two spherical surfaces are to each other as the squares of the radii.* Prove.

3. How many square feet should be allowed for polishing a hemispherical dome whose diameter is $1\frac{3}{4}$ feet?

4. The earth is approximately a sphere of diameter equal to 7,920 miles. How large is its surface?

5. Find the area of the north temperate zone, assuming its altitude to be about 1,800 miles.

6. Show that the surface of a sphere is equal to the lateral surface of the circumscribed cylinder.

7. Find the ratio of the area of the surface of the moon to that of the earth assuming the diameter of the moon to be 2,162 miles.

8. Two parallel planes, equidistant from the center of a sphere of radius r, cut from the sphere a zone whose area is $\frac{5}{4}$ the area of the curved surface of the cylinder having the same bases as the zone. Find the distance of the planes from the center of the sphere. (Harvard.)

9. How far in one direction can a man see from the top of Mount Etna?

The required distance is the geometric mean between the height of the mountain, 3,300 m., and the sum of the height and the diameter of the earth, 6,374 kilometers.

10. Show that if a man ascended in a balloon to a height equal to the earth's radius, he would see one-quarter of the earth's surface. (Harvard.)

11. The lateral area of a cone of revolution and the area of a sphere are each equal to 49 square feet. If the radius of the sphere equals the radius of the base of the cone, find the altitude of the cone.

12. The eight vertices of a cube all lie on a sphere. Prove that every diagonal of the cube is a diameter of the sphere. If one edge of the cube is a, find the area of the zone of one base cut off by the plane of one face of the cube. (Harvard.)

13. If the temperate zones were between the 30° and 60° parallels of latitude, what proportion of the earth's surface would they comprise? Give the details of the computation. (Board.)

14. Two parallel planes on the same side of the center of a sphere of radius r bound a zone. The area of this zone is one-fourth that of the sphere. The area of the circle cut by the plane nearer to the center is double that cut by the farther. Find the distance from the center of the sphere to the nearer plane. (Harvard.)

282. The chapter has taught the meaning of the following terms:

polyedron
face, edge, vertex, surface of a polyedron
tetraedron, hexaedron
octaedron, dodecaedron, icosaedron
pyramidal and conical surface
directrix, generatrix
triangular, quadrangular, pentagonal pyramid
regular pyramid
slant height
circular, right circular cone
cone of revolution
cylindrical surface, cylinder
right cylinder, oblique cylinder
cylinder of revolution
prismatic surface, prism
right and oblique prism
triangular, quadrangular, etc., prism
section, right section
parallelopiped
truncated prism
frustum of a pyramid
frustum of a cone
sections of a right circular cone
circle, ellipse, parabola, hyperbola
similar cylinders, similar cones
surface of revolution
zone

Summary

283. The truth of the following theorems has been established:

1. The equation $f+v=e+2$ expresses the relation between the number of faces, vertices, and edges of a convex polyedron.

2. *The lateral edges of a regular pyramid are equal.*

3. *The lateral faces of a regular pyramid are congruent isosceles triangles.*

4. *The lateral edges of a prism are equal.*

5. *The lateral faces of a prism are parallelograms.*

6. *The sections of a prism made by parallel planes are congruent.*

7. *The right sections of a prism are congruent.*

8. *A section of a prism parallel to the base is congruent to the base.*

9. *The section of a cylinder made by a plane passing through an element is a parallelogram.*

10. *The sections of a cylinder made by parallel planes cutting all elements are congruent.*

11. *The sections of a cylinder parallel to the bases are congruent to the base.*

12. *A section of a cone made by a plane passing through the vertex is a triangle.*

13. *A section of a circular cone made by a plane parallel to the base is a circle.*

14. *If a pyramid is cut by a plane parallel to the base, the edges and altitude are divided proportionally; the section is a polygon similar to the base; the areas of the section and the base are proportional to the squares of the distances from the vertex.*

15. *The lateral areas, or the total areas of similar circular cylinders, or cones, are proportional to the squares of the altitudes, or to the squares of the radii of the bases.*

16. The plane sections of a right circular cone are two intersecting straight lines, a circle, a parabola, an ellipse, and a hyperbola.

284. The following is a summary of the formulas of this chapter:

I. Lateral area:

1. Of a prism,

$$L = p \times e$$

2. Of a right prism,

$$L = p \cdot h$$

3. Of a regular pyramid,

$$L = \tfrac{1}{2} s \times p$$

4. Of a frustum of a pyramid,

$$L = \tfrac{1}{2}(p_1 + p_2)s$$

5. Of a right cylinder,

$$L = 2\pi r h$$

6. Of a right cone,

$$L = \pi r \cdot s$$

7. Of a frustum of a right circular cone,

$$L = \pi(r_1 + r_2)s$$

II. The area of a surface generated by revolving half of a regular polygon about a diagonal joining two directly opposite vertices,

$$L = 2\pi \overline{MO} \times \overline{AF}$$

III. The area of the surface of a sphere,

$$S = 4\pi r^2$$

IV. The area of a zone,

$$Z = 2\pi r h$$

CHAPTER XIII

VOLUMES

Volume of a Rectangular Parallelopiped

285. Volume. To measure the space bounded by the surface of a solid, a cube is used whose edges are the unit of length. This cube is said to be the **unit of volume,** and the number of times it is contained in the solid is the **volume** of the solid.

286. Volume of a rectangular parallelopiped. Let a, b, and c be the lengths of three concurrent edges, Fig. 171. By drawing planes parallel to the base, the parallelopiped, p, may be divided into a equal layers (l), Fig. 172. Each layer, l,

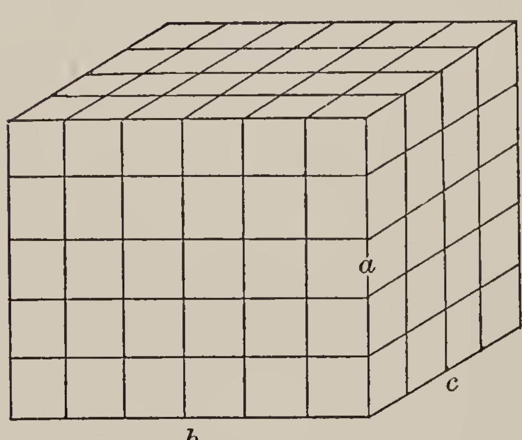


Fig. 171

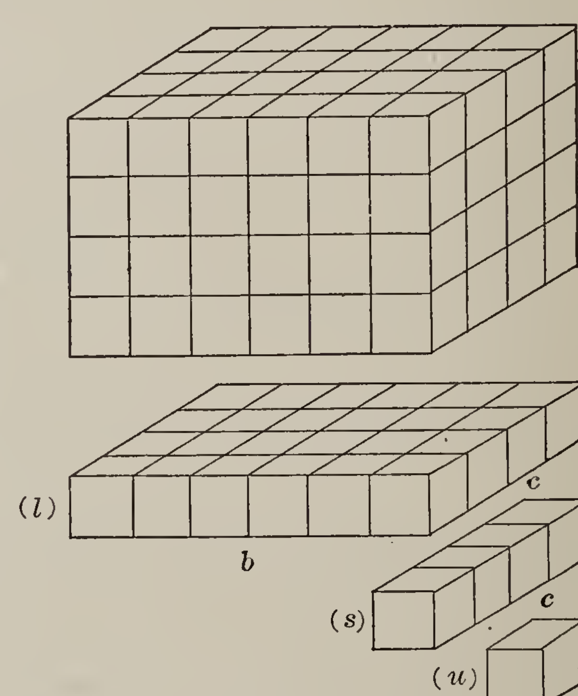


Fig. 172

may be divided into b equal strips, (s), and each strip, s, into c equal cubes (u.)

Hence the volume of a strip s is c, the volume of a layer l is $b \times c$ and the volume of the parallelopiped, p is $a \times b \times c$.

So far we have assumed that a, b, and c are commensurable. Let us suppose a, b, and c to be incommensurable, e.g., $a = \sqrt{6}$m; $b = \sqrt{10}$m; $c = \sqrt{15}$m. Then $a = 2.4494 \ldots$ m, $b = 3.1623 \ldots$ m, $3c = .8730 \ldots$ m. In this case the volume of the parallelopiped may be determined to any desired degree of accuracy, as follows:

1. Taking $a = 2.4$, $b = 3.1$, and $c = 3.8$ and using 1 decimeter as the unit of length, we have $V = 24 \times 31 \times 38$ cubic decimeters $= 28{,}272$ cubic decimeters $= 28.272$ cubic meters.

2. Similarly, for $a = 2.44$, $b = 3.16$, and $c = 3.87$, using a centimeter as unit, $V = 244 \times 316 \times 387$ cubic centimeters $= 29{,}993{,}456$ cubic centimeters $= 29.839248$ cubic meters.

3. For $a = 2.449$, $b = 3.162$, and $c = 3.873$, using a millimeter as unit of length, $V = 29{,}991{,}497{,}274$ cubic millimeters $= 29.9914 \ldots$ cubic meters.

4. For $a = 2.4494$, $b = 3.1623$, and $c = 3.8730$ we find $V = 29{,}999{,}241{,}802{,}260$ cubic one-tenth millimeters $= 29.9992 \ldots$ cubic meters.

By taking a, b, and c to a still greater number of decimal places, we may obtain an approximate value of V differing from the actual value by a number less than any assigned quantity.

Assuming the formula $V = a \cdot b \cdot c$ to hold for *incommensurable* values of a, b, and c, we find

$$\begin{aligned} V &= \sqrt{6} \times \sqrt{10} \times \sqrt{15} \text{ cubic meters} \\ &= \sqrt{6 \times 10 \times 15} \text{ cubic meters} = 30 \text{ cubic meters.} \end{aligned}$$

However, this is the value approached by the sequence

$$V = 28.272,\ 29.8392,\ 29.9914,\ 29.9992,\ \text{etc.}$$

Thus, whether a, b, and c have commensurable or incommensurable values, the preceding discussion shows that *the volume of a rectangular parallelopiped is equal to the product of the three dimensions.*

EXERCISES

Prove the following:

1. *The volume of a rectangular parallelopiped is equal to the product of the base by the altitude.*

2. *The volume of a cube is equal to the cube of an edge.*

3. *The volumes of two cubes are to each other as the cubes of the edges.*

4. *Two rectangular parallelopipeds are to each other as the products of the three dimensions.*

5. *Two rectangular parallelopipeds, having equal altitudes (bases) are to each other as the bases (altitudes).*

6. *Two rectangular parallelopipeds having two (one) dimensions equal are to each other as the third (product of the other two) dimension.*

7. Show that the volume of a cube varies directly as the cube of the edge. If the edge of a given cube is doubled, trebled, etc., how does the volume of the new cube compare with that of the given cube?

8. How many dimensions are needed to determine the volume of a parallelopiped? The area of one face?

9. A room is 12.5 ft. long, 12 ft. wide, and 11 ft. high. Find how many cubic feet of air it contains.

10. How many bricks will be needed to build a wall $20 \times 4 \times 2$ ft., making no allowance for mortar?

Assume the size of a brick to be $9 \times 3 \times 4\frac{1}{2}$ inches.

Comparison of Volumes

287. Theorem: *The plane passed through two diagonally opposite edges of a right parallelopiped divides the parallelopiped into two equal triangular right prisms.*

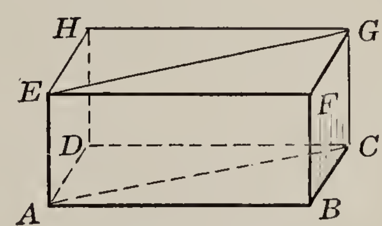


Fig. 173

Given the right parallelopiped AG, Fig. 173, plane $ACGE$ passed through AE and CG.

To prove that prism

$$ABC\text{–}F \cong \text{prism } CDA\text{–}H.$$

Proof: $\triangle ABC \cong \triangle CDA$. Why?

Imagine ABC–F placed on CDA–H making $\triangle ABC$ coincide with $\triangle CDA$.

Then BF will coincide with DH, AE with GC, and GC with AE. Why?

Hence $\triangle EFG$ will coincide with $\triangle GHE$. Why?

$\therefore$ Prisms ABC–F and CDA–H coincide throughout and are congruent.

288. Theorem: *An oblique prism is equal to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*

Given the oblique prism AD', Fig. 174, and the right prism FI', FI being a right section of prism AD'; $FF' = AA'$.

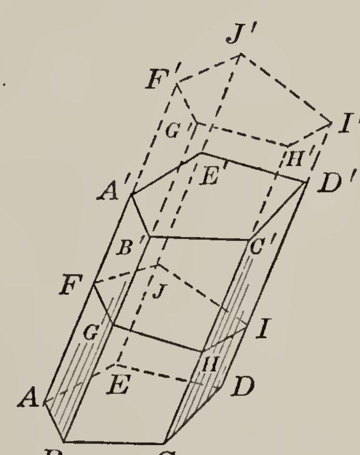


Fig. 174

To prove that $AD' = FI'$.

Proof: Imagine the truncated prism AI to be placed on the truncated prism $A'I'$ making AD coincide with $A'D'$.

Show that FA, GB, HC, etc., coincide respectively with $F'A'$, $G'B'$, $H'C'$, etc.

Show that AG, BH, etc., coincide respectively with $A'G'$, $B'H'$, etc.

Hence $AI = A'I'$, since they can be made to coincide.

But $\underline{AI' \equiv AI'}$

$\therefore$ $AD' = FI'$

(Equals subtracted from equals give equals.)

Exercises

1. The diagonal of a cube is $10\sqrt{3}$. Find the volume.

2. Find the surface and volume of a cube whose diagonal is 24 inches.

3. How many gallons of water are contained in a tank whose shape is that of a rectangular parallelopiped whose dimensions are 12, 20, and 10.4 feet respectively?

A gallon contains 231 cubic inches.

4. Given a sphere whose diameter is 10 inches. Find the volume and the surface of the inscribed cube. (Sheffield.)

Volume of a Prism

289. Theorem: *The volume of a right triangular prism is equal to the product of the base by the altitude.*

Given the right triangular prism $ABC\text{-}F$, Fig. 175.

To prove that

$$ABC\text{-}F = \overline{ABC} \times \overline{CF}.$$

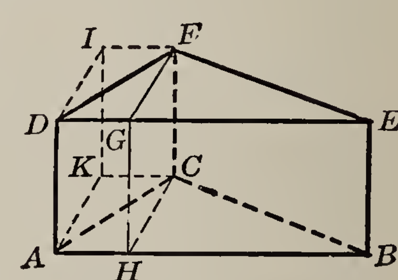


Fig. 175

Proof: Draw $FG \perp DE$ and $CH \perp AB$.

Show that FG and CH are both perpendicular to plane AE.

Pass a plane through FG and CH.

Draw $AK \parallel HC$, $CK \parallel HA$. Draw $KI \parallel HG$, meeting plane DGF in I.

Then $AHCK\text{–}F$ is a rectangular parallelopiped.

$$AHC\text{–}F = \tfrac{1}{2}\, AHCK - F \quad (\S\, 287).$$

$$\therefore\ AHC\text{–}F = \tfrac{1}{2}\, AHCK \times CF \quad (\S\, 286).$$

$$\therefore\ AHC\text{–}F = \overline{AHC} \times \overline{CF}. \qquad \text{Why?}$$

Similarly, prove that

$$HBC - F = \overline{HBC} \times \overline{CF}.$$

Adding, $ABC\text{–}F = \overline{ABC} \times \overline{CF}.$

290. Theorem: *The volume of a right parallelopiped is equal to the product of the base by the altitude.*

Divide the parallelopiped into two equal right triangular prisms (§ 287). Then find the sum of the two triangular prisms.

291. Theorem: *The volume of an oblique parallelopiped is equal to the product of the base by the altitude.*

Given the oblique parallelopiped AG, Fig. 176, whose base is $ABCD$ and whose altitude is h.

To prove that $AG = ABCD \times h$.

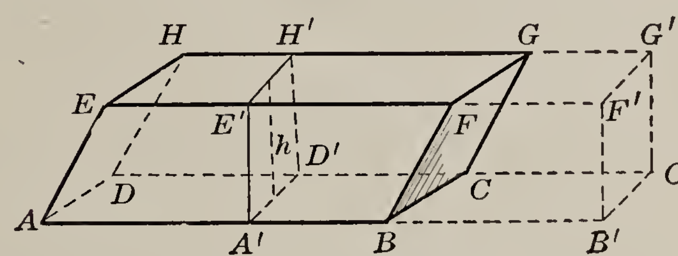


FIG. 176

Proof: Construct the right section $A'H'$.

On $A'H'$ as base construct the right parallelopiped $A'G'$, having its edge $A'B'$ equal to AB.

Then $\qquad AG = A'G'. \qquad$ Why?

But

$$\begin{aligned} A'G' &= A'H' \times A'B' \\ &= (h \times A'D') \times A'B' \\ &= h \times A'D' \times AB \\ &= h \times (A'D' \times AB) \\ &= h \times ABCD. \end{aligned}$$

Briefly, this result may be expressed by the equation

$$\boldsymbol{V = h \cdot b.}$$

292. Theorem: *The plane passed through two diagonally opposite edges of any parallelopiped divides the parallelopiped into two equal triangular prisms.*

Draw the right section $IJKL$, Fig. 177.

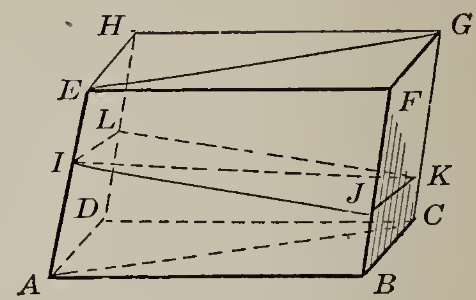


Fig. 177

Show that the triangular prism $ABC\text{–}F =$ the right prism having the base IJK and the altitude equal to BF.

Show that $CDA\text{–}H =$ the right prism having the base KLI and the altitude equal to BF.

Show that these two right prisms are equal to each other (§ 289).

$\therefore \quad ABC\text{–}F = CDA\text{–}H.$

293. Theorem: *The volume of any triangular prism is equal to the product of the base by the altitude.*

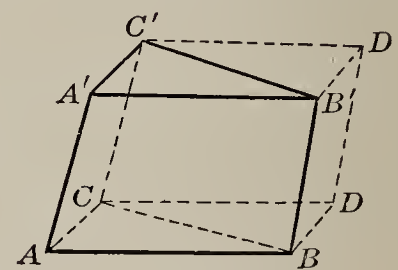


Fig. 178

Construct the parallelogram $ABCD$, Fig. 178.

Construct the parallelopiped $ABDC\text{–}B'$.

Then $\quad ABC\text{–}B' = \frac{1}{2}ABDC\text{–}B' \quad$ (§ 292).

$ABDC\text{–}B' = ABDC \times h \quad$ (§ 291).

$\therefore \quad ABC\text{–}B' = \frac{1}{2}ABDC \times h$

$= ABC \times h.$

294. Theorem: *The volume of any prism is equal to the product of the base by the altitude.*

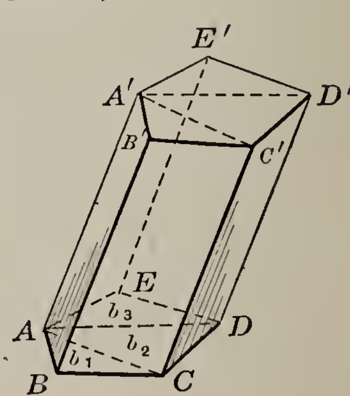


Fig. 179

By drawing planes through AA', Fig. 179, and all the other lateral edges, the prism may be divided into triangular prisms, having the same altitude, h, and bases b_1, b_2, b_3, etc.

Then
$$ABC-A' = b_1h,$$
$$ACD-A' = b_2h,$$
$$ADE-A' = b_3h, \text{ etc.}$$

Adding, $ABCDE-A' = (b_1+b_2+b_3+ \text{ etc.})h.$

This result may be expressed briefly by means of the equation
$$V = b \cdot h.$$

EXERCISES

1. *Prisms having equal bases and altiiudes are equal.* Prove.

2. *The volumes of two prisms having equal bases are to each other as the altitudes.* Prove.

3. How many cubic yards of earth must be removed to build a trench 200 ft. long and 10 ft. deep, 4 ft. wide at the bottom and 6 ft. at the top?

4. What will be the cost, at 42 cents a cubic yard, to dig a ditch 10 rd. long, 4 ft. deep, 7 ft. wide at the top, and $4\frac{1}{2}$ ft. wide at the bottom?

5. Find the volume of a right triangular prism, 8 in. high, whose base is an equilateral triangle with sides of 2 inches.

6. A triangular prism is 10 ft. high. The base is a right triangle whose sides are 3, 4, and 5 ft. respectively. Find the volume.

7. The volume of a triangular prism is 250. The base is an equilateral triangle whose side is 7. Find the altitude of the prism.

8. Find the volume of a triangular prism whose height is 30 in. and the sides of whose base are 12 in., 10 in., and 10 inches.

9. Find the volume of a prism whose base is a rhombus, one of whose sides is 40 in., and whose shorter diagonal is 48 inches. The height of the prism is 60 inches.

10. A regular hexagonal prism has the area of one base 12 and its total area 276. Find the volume of the prism. (Yale.)

295. Inscribed prism, pyramid, and frustum of a pyramid. A prism, a pyramid, and a frustum of a pyramid are said to be **inscribed** if the lateral edges are elements of a cylinder, a cone, or a frustum of a cone, respectively, and if the bases of the former are inscribed in the bases of the latter, Fig. 180. It will be *assumed* that

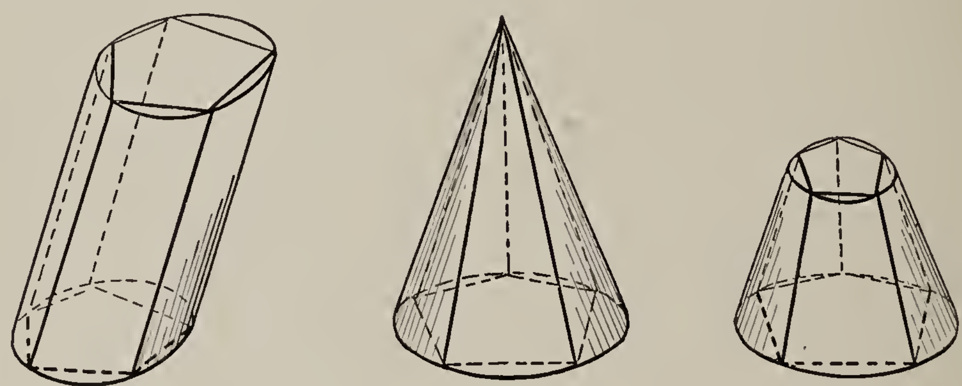


FIG. 180

the volume of an inscribed prism, pyramid, or frustum of a pyramid is less than the volume of the cylinder, cone, and frustum of a cone respectively.

296. Tangent plane. If a plane contains one, and only one, element of a cylinder, a cone, or a frustum of a cone, but does not intersect the surface, it is a **tangent plane**, Fig. 181.

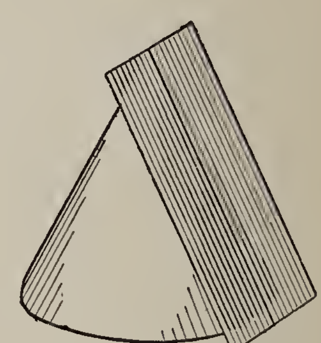


FIG. 181

EXERCISES

1. Prove that a plane passing through a tangent to the base of a circular cone and the element drawn through the point of contact is tangent to the cone.

Is this theorem necessarily true when the cone is not circular? (Harvard.)

2. The intersection of two planes tangent to a circular cylinder is parallel to the elements of the cylinder. (Yale.)

297. Circumscribed prism, pyramid, and frustum of a pyramid. A prism, a pyramid, and a frustum of a pyramid are said to be **circumscribed** if the lateral faces are tangent to the lateral surface of a cylinder, a cone, and a frustum of a cone, respectively, and if the bases of the former are circumscribed about the bases of the latter, Fig. 182. It will be *assumed* that the volume of a circum-

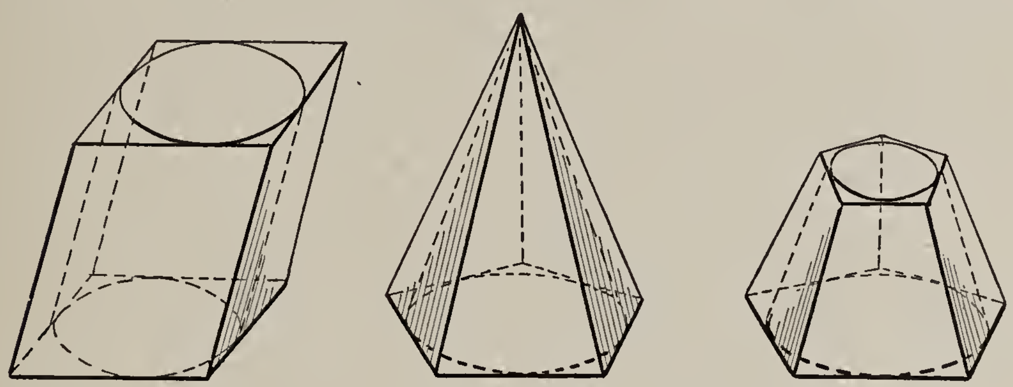


FIG. 182

scribed prism, pyramid, or frustum of a pyramid is greater than the volume of the cylinder, cone, or frustum of a cone respectively.

Volume of a Cylinder

298. Theorem: *The volume of a circular cylinder is equal to the product of the base by the altitude.*

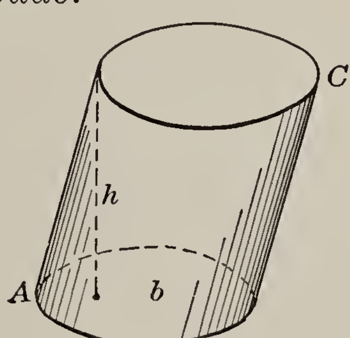


FIG. 183

Given the cylinder AC, Fig. 183, whose altitude is h and whose base is b.

To prove that the volume, $\boldsymbol{v = b \cdot h}$.

Proof (indirect method):

Assume that $v \neq b \cdot h$. Then $v > bh$, or $v < bh$.

First, assume $v < bh$, or $v = Bh$, where $b > B$.

Inscribe in the cylinder a prism whose base, B', is greater than B, i.e., such that $b > B' > B$.

Then $B'h > Bh$.

Thus $B'h$, the volume of the *inscribed* prism, is *greater* than Bh, the volume of the *cylinder*.

This is impossible, and v is not less than $b \cdot h$.

Secondly, assume $v > bh$, or $v = Bh$, where $b < B$.

Circumscribe about the cylinder a prism whose base, B', is less than B, i.e., such that $b < B' < B$.

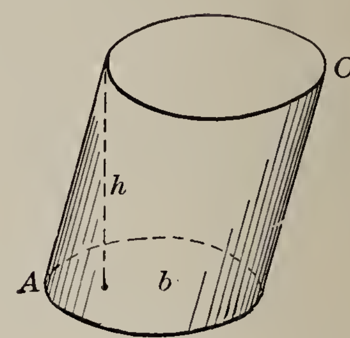


FIG. 183

Then $B'h < B \cdot h$.

Thus the volume of the *circumscribed* prism is *less* than the volume of the *cylinder* which was assumed to be equal to $B \cdot h$.

This is impossible, and v is not greater than $b \cdot h$.

$$\therefore \quad v = b \cdot h.$$

299. Theorem: *The volume of a circular cylinder of revolution is given by the formula*

$$V = \pi r^2 \cdot h$$

where h is the altitude and r the radius of the base. Prove.

EXERCISES

1. How many cubic yards of dirt must be removed in the excavation of a tunnel 20 ft. high and $\frac{3}{4}$ mi. long? The shape of the tunnel is to be such as to make the cross-section a semicircle.

2. A cubic foot of copper is drawn into a wire $\frac{1}{8}$ in. in diameter. What is the length of the wire?

3. The inner diameter of a pipe 75 yd. long is $2\frac{3}{4}$ inches. How many feet of water does it contain?

4. The cost of digging a well is $3.25 per cubic yard. What will it cost to dig a well 80 ft. deep and 5 ft. in diameter?

5. Find the volume of a circular ring whose inner radius is 8 in. and whose outer radius is 10 inches.

Consider the ring as a cylinder whose height is the length of a circle whose radius is the mean of the inner and outer radii.

6. What must be the height of a water boiler holding 30 gal., if its diameter is 1 foot?

7. The altitude of a cylinder of revolution is 26. The radius of the base is 24. Find the volume.

8. *The volumes of two similar cylinders of revolution are to each other as the cubes of the altitudes, or as the cubes of the radii of the bases.* Prove.

Show that $\frac{V}{V'} = \frac{\pi r^2 h}{\pi r'^2 h'} = \frac{r^2}{r'^2} \times \frac{h}{h'} = \frac{r^2}{r'^2} \times \frac{r}{r'} = \frac{r^3}{r'^3}$.

9. A cylindrical tank, Fig. 184, is partly filled with water. The tank is 4 ft. long and 10 in. in diameter. If the greatest depth of the water is 8 ft., how much of the tank is filled with water?

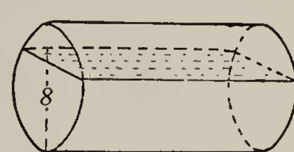


FIG. 184

10. A rectangle is rotated about one of its sides as an axis. What is the ratio of the volumes generated by the triangles into which the rectangle is divided by one of its diagonals? (Board.)

Use § 306.

11. A cube, 5 in. on a side, is just covered by water in the bottom of a cylindrical pail 12 in. in diameter. How high will the water stand in the pail when the cube is removed? (Harvard.)

12. Compare the volumes of cylinders of altitude 8 in. and 10 in., respectively, whose convex surface can be exactly covered by a rectangular sheet of paper 8 by 10 in. in size. (Yale.)

13. Prove that if a square be rotated completely around a straight line in its plane, not crossing it, but parallel to one of its sides, the volume generated is equal to the product of the area of the square and the length of the circumference traced by its center. (Harvard.)

14. A glass vessel made in the form of a right circular cylinder contains a certain amount of water. The diameter of the base of the vessel is 5 inches. When an irregular mass of gold is dropped into the vessel it is entirely covered by water and the level of the water rises 3 inches. What is the weight in ounces of the lump of gold if gold weighs 11 oz. per cubic inch? (Board.)

15. A regular hexagonal prism is inscribed in a right circular cylinder whose base has radius 10. Compare their lateral areas and their volumes. (Yale.)

Volume of a Pyramid

300. Theorem: *If two pyramids have equal bases and equal altitudes, sections made by planes parallel to the bases and at equal distances from the vertices are equal.*

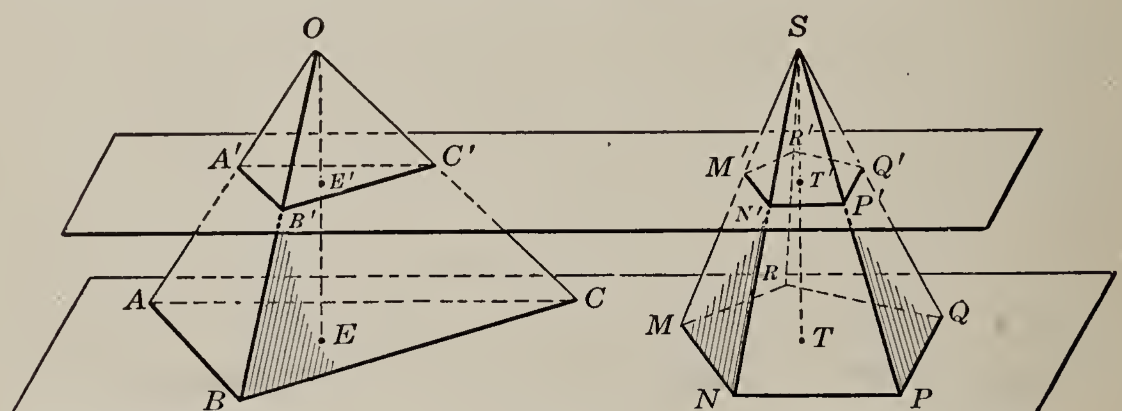


FIG. 185

Proof:

$$\frac{A'B'C'}{ABC}=\frac{\overline{OE'}^2}{\overline{OE}^2} \qquad (\S\ 260)$$

$$\frac{M'N'P'Q'R'}{MNPQR}=\frac{\overline{ST'}^2}{\overline{ST}^2}$$

$$\frac{\overline{OE'}^2}{\overline{OE}^2}=\frac{\overline{ST'}^2}{\overline{ST}^2} \qquad \text{Why?}$$

$$\therefore \frac{A'B'C'}{ABC}=\frac{M'N'P'Q'R'}{MNPQR} \qquad \text{Why?}$$

$$\therefore A'B'C'=M'N'P'Q'R' \qquad \text{Why?}$$

301. Theorem: *If two triangular pyramids have equal bases and altitudes, their volumes are equal.*

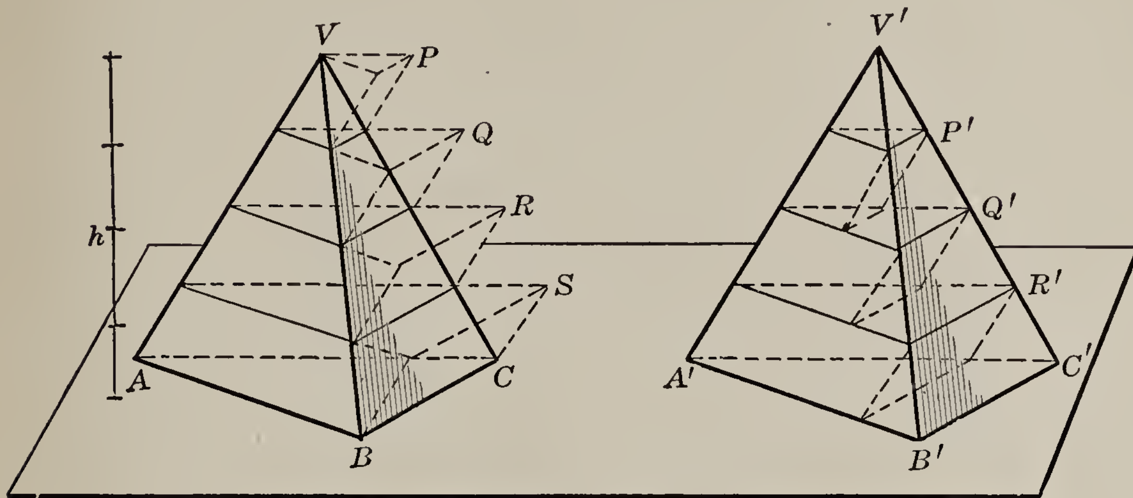


Fig. 186

Discussion: Place the bases of the pyramids, Fig. 186, in the same plane and divide the altitude h into equal parts, x.

Through the points of division draw planes parallel to the plane of the bases, cutting the pyramids into sections equal in pairs (§ 300).

Using each section as *lower base,* construct prisms whose altitudes are equal to x and whose lateral edges are parallel to AV and $A'V'$ respectively. In this way the prisms P, Q, R, and S have been constructed in pyramid V–ABC. Imagine a similar set of prisms constructed in V'–$A'B'C'$.

Using each section as *upper base,* construct prisms whose altitudes are equal to x and whose lateral edges are parallel to AV and $A'V'$ respectively. This gives prisms P', Q', and R' in the pyramid V'–$A'B'C'$. Imagine a similar set of prisms constructed in V–ABC.

Show that $P=P'$, $Q=Q'$, $R=R'$.

Denote $P+Q+R+S$ by X and $P'+Q'+R'$ by Y.

Then $X-Y=S$.

Denoting the volumes of $V\text{–}ABC$ and $V'\text{–}A'B'C'$ by V and V', respectively, we have

$$X > V > Y$$

and $$X > V' > Y.$$

$\therefore$ the difference between V and V' is less than the difference between X and Y,

i.e., $$V - V' < X - Y,$$

or $$V - V' < S.$$

By dividing the altitude h into twice as many parts as before, a new set of prisms is obtained for which the same relations hold and $V - V' < S_1$, where S_1 is half as large as S. This process can be repeated as often as required, and an S_n obtained as small as anyone may assign and such that $V - V' < S_n$.

To show that $V = V'$ we may proceed as follows:

Assume $V \neq V'$, let $V > V'$,
and denote the difference by d,

$$\text{i.e., } V - V' = d.$$

We have seen that by increasing the number of divisions in the altitude h, we can make S less than any assigned quantity, therefore less than d.

$$\text{Hence, } V - V' < d.$$

This contradicts the statement

$$V - V' = d.$$

Therefore the assumption $V \neq V'$ is wrong and $V = V'$.

BONAVENTURA CAVALIERI

BONAVENTURA CAVALIERI

BONAVENTURA CAVALIERI, professor of mathematics at Bologna, was born at Milan in 1598 and died at Bologna in 1647. He was one of the most influential mathematicians of his day. Through his influence logarithms were introduced into Italy. He discovered the law for the area of a spherical triangle in terms of its spherical excess. His chief claim to renown rests on his clear enunciation in 1629 of the *principle of indivisibles.* This principle, first published by Cavalieri in 1635, though unsound philosophically, is of great mathematical significance because it became the progenitor of the infinitesimal calculus.

Cávalieri's statement of the principle in 1635 said a line was made up of an infinite number of points, each without magnitude, a surface of an infinite number of lines, each without breadth, and a volume of an infinite number of surfaces, each without thickness. This form of statement was vigorously objected to by contemporary mathematicians, and it was later correctly stated that the method of indivisibles rests on the assumption that any magnitude may be divided into an infinite number of small quantities, which can be made to bear any required ratios to one another.

The method of indivisibles in turn grew out of the tedious method of exhaustions, used by the ancient Greeks. In the eighteenth century the integral calculus replaced the method of indivisibles..

One may see how the method of indivisibles was used by Cavalieri from two examples given on pp. 280 and 281 of Ball.

302. Cavalieri's* theorem. Let V and V', Fig. 187, be two solids lying between parallel planes M and N,

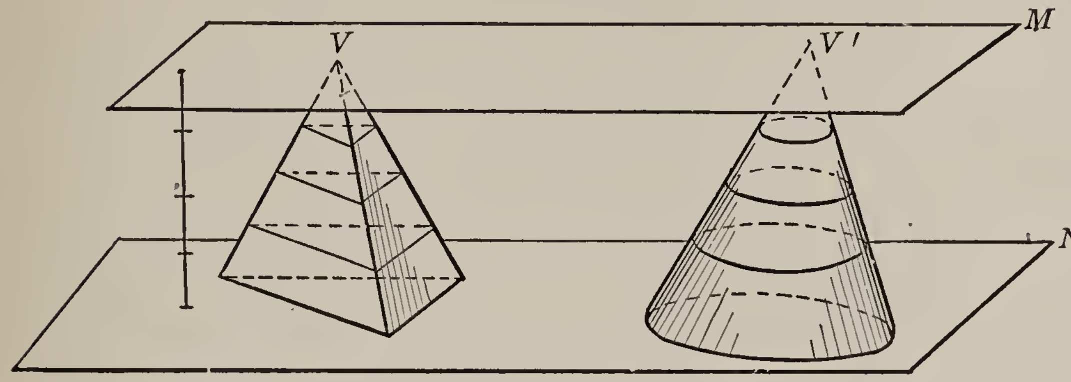


Fig. 187

and let the two sections cut from V and V' by any plane parallel to M and N be equal.

By dividing the distance between M and N into equal parts and by drawing planes through the points of division parallel to N, the solids may be divided into slices.

As the number of planes between M and N is increased, the slices become approximately prismatic or cylindrical.

Any two corresponding slices are equal, since they have equal bases and altitudes, § 294.

Hence the sum of all slices of V is equal to the sum of all slices of V'.

By increasing indefinitely the number of slices, and by a process of reasoning similar to that of § 301, it may be seen that $V = V'$.

This fact is known as **Cavalieri's theorem** and may be stated as follows: *If two solids lie between two given parallel*

* Bonaventura Cavalieri was born in Milan in 1598 and died in Bologna in 1647. He gained a reputation through his "principle of indivisibles" in which he asserted that lines were made up of an infinite number of points, surfaces of an infinite number of lines, and solids of an infinite number of planes. Though unscientific, Cavalieri's method was used for years as a sort of integral calculus. See Cajori, p. 171, and Ball, pp. 279 and 280.

planes, having their bases in these planes, and if the sections made by any plane parallel to the given planes are equal, then the volumes of the solids are equal.

303. Theorem: *The volume of a triangular pyramid is equal to one-third the product of the base by the altitude.**

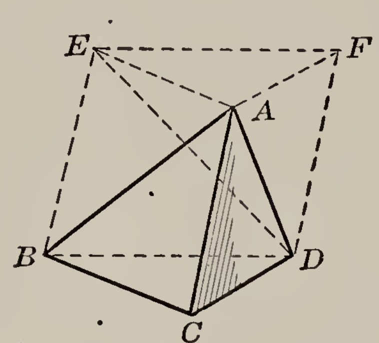


FIG. 188

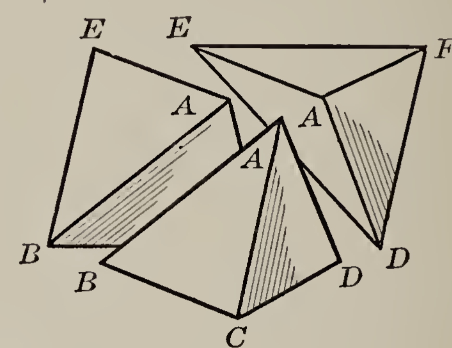


FIG. 189

Given the triangular pyramid $A-BCD$, Fig. 188, with volume v, base b, and altitude h.

To prove that $v=\frac{b \cdot h}{3}$.

Proof: On BCD as base construct the triangular prism $BCD-E$ having the altitude h, and AC as one of the lateral edges.

Planes EAD and BAD divide this prism into three triangular pyramids, as shown in Fig. 189.

Then $A-BCD=D-EAF$ (§ 301),

$D-EAF \equiv A-EFD$

$A-EFD=A-BED$ Why?

Hence prism $BCD-E$ has been divided into three equal triangular pyramids.

$\therefore$ the pyramid $A-BCD=\frac{1}{3}$ of the prism $BCD-E$,

or $$v=\frac{1}{3}b \cdot h.$$

* This theorem was demonstrated by Eudoxus (b. 408 B.C.).

304. Theorem: *The volume of any pyramid is equal to one-third the product of the base by the altitude i.e.,*

$$v=\frac{1}{3}b\cdot h.$$

By passing planes through the vertex V, Fig. 190, and the diagonals of the base, the given pyramid may be divided into triangular pyramids $V-AEB$, $V-BEC$, etc. The sum of the volumes of these triangular pyramids is the volume of the given pyramid.

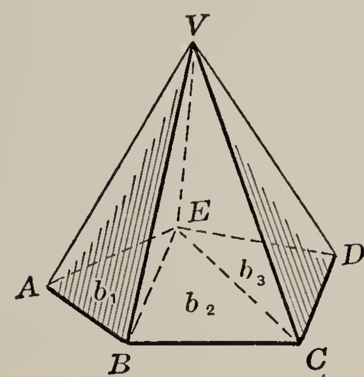


FIG. 190

Thus $\quad v=\frac{1}{3}hb_1+\frac{1}{3}hb_2+\text{etc.}=\frac{1}{3}h(b_1+b_2+\text{etc.}).$

$$\therefore \quad v=\frac{1}{3}hb,$$

where h is the length of the altitude and b the area of the base of the pyramid V–$ABCDE$.

EXERCISES

1. Find the volume of a pyramid whose base is a square 10 in. long and whose altitude is 12 inches.

2. The edges of a regular tetraedron are 3 in. long. Find the volume.

3. The base of a regular pyramid is a square each side of which is 4 feet. The pyramid is 5 ft. high. Find the volume.

4. Find the volume of a regular hexagonal pyramid, the perimeter of whose base is 12 inches. The pyramid is 5 in. high.

5. Originally the great pyramid of Cheops was 480 ft. 9 in. high and the side of the square base was 764 ft. long. Owing to the removal of coating the measurements are now 746 ft. and 460 ft. respectively. How much stone has been removed?

6. The area of the base of a regular quadrangular pyramid is 400. The altitude is 10. Find the volume. How far from the vertex is a section parallel to the base whose area is 100?

7. The base of a regular pyramid is a regular hexagon which can be inscribed in a circle of radius 10. One of the lateral edges of the pyramid is 20. Find the volume of the pyramid. (Harvard.)

8. Three edges of a parallelopiped are AB, AC, and AD. Prove that the plane BCD divides the parallelopiped into two solids whose volumes are in the ratio of five to one.

In what ratio are the volumes of the two solids into which the parallelopiped is divided by the plane bisecting these three edges? (Harvard.)

Volume of a Frustum of a Pyramid

305. Denote the upper and lower base of the frustum of a pyramid, Fig. 191, by b_1 and b_2, respectively, and their distances from the vertex by h_1 and h_2.

Then $$\frac{b_2}{b_1}=\frac{h_2^2}{h_1^2} \quad (\S\ 260).$$

Denoting the altitude O_1O_2 of the frustum by h, we have $$h_1=h_2-h.$$

$$\therefore \quad \frac{b_2}{b_1}=\frac{h_2^2}{(h_2-h)^2}$$

$$\therefore \quad \frac{\sqrt{b_2}}{\sqrt{b_1}}=\frac{h_2}{h_2-h}$$

$$\therefore \quad h_2\sqrt{b_2}-h\sqrt{b_2}=h_2\sqrt{b_1}$$

$$\therefore \quad h_2\sqrt{b_2}-h_2\sqrt{b_1}=h\sqrt{b_2}$$

$$\therefore \quad h_2=\frac{h\sqrt{b_2}}{\sqrt{b_2}-\sqrt{b_1}}$$

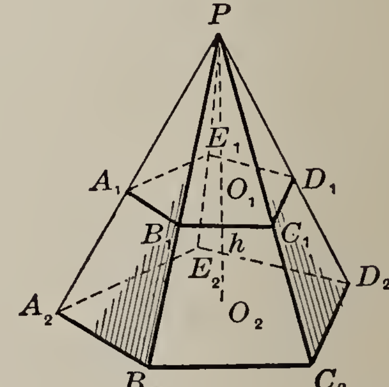


FIG. 191

Denote the volume of the frustum by v.

Since the frustum

$$A_2D_1=(P-A_2B_2C_2D_2E_2)-(P-A_1B_1C_1D_1E_1,)$$

we have $v=\frac{1}{3}b_2h_2-\frac{1}{3}b_1h_1=\frac{1}{3}b_2h_2-\frac{1}{3}b_1(h_2-h)$

$$=\frac{1}{3}b_2h_2-\frac{1}{3}b_1h_2+\frac{1}{3}b_1h$$
$$=\frac{1}{3}b_1h+\frac{1}{3}h_2(b_2-b_1)$$
$$=\frac{1}{3}b_1h+\frac{1}{3}\frac{h\sqrt{b_2}(b_2-b_1)}{\sqrt{b_2}-\sqrt{b_1}}$$
$$=\frac{1}{3}b_1h+\frac{1}{3}\frac{h\sqrt{b_2}(\sqrt{b_2}-\sqrt{b_1})(\sqrt{b_2}+\sqrt{b_1})}{\sqrt{b_2}-\sqrt{b_1}}$$
$$=\frac{1}{3}b_1h+\frac{1}{3}h\sqrt{b_2}(\sqrt{b_2}+\sqrt{b_1})$$
$$=\frac{1}{3}b_1h+\frac{1}{3}h(\sqrt{b_2b_2}+\sqrt{b_1}\sqrt{b_2})$$
$$\therefore\quad \boldsymbol{v=\frac{1}{3}h(b_1+b_2+\sqrt{b_1b_2})}$$

EXERCISE

The stone cap of a gatepost is in the form of a regular square pyramid whose base measures 4 in. on a side and whose altitude is 15 inches. If the top of the cap is cut off by a plane parallel to its base and 5 in. above it, what is the volume of the piece cut off? (Board.)

Volume of a Circular Cone

306. Theorem: *The volume of a circular cone is equal to one-third the product of the base by the altitude,*

i.e., $\boldsymbol{v=\frac{1}{3}b\cdot h.}$

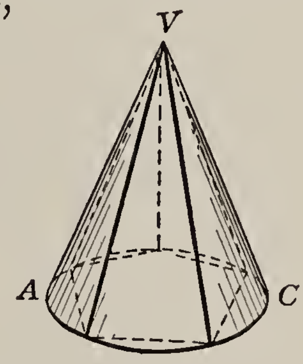


FIG. 192

Proof (indirect method):

1. Suppose $v<\frac{1}{3}bh$

and that $v=\frac{1}{3}Bh$, where $B<b$.

Inscribe a pyramid, Fig. 192, whose base $B'>B$.

Then $\frac{1}{3}B'h>\frac{1}{3}Bh$.

This means that the volume of the inscribed pyramid is greater than the volume of the cone.

This is impossible, and v is not less than $\frac{1}{3}bh$.

2. Similarly we may show that v is not greater than $\frac{1}{3}bh$.

3. Hence $\boldsymbol{v=\frac{1}{3}b\cdot h.}$

EXERCISE

A right triangle is revolved about one leg. Show that the volume of the cone thus generated is equal to the product of the area of the triangle and the circumference of the circle traced by the point of intersection of the medians. (Harvard.)

Volume of a Frustum of a Cone

307. The formula giving the volume of the **frustum of a cone** is obtained in the same way as the formula for the volume of the frustum of a pyramid.

Hence, $$v=\frac{1}{3}h(b_1+b_2+\sqrt{b_1b_2}).$$

308. To find the volume of a **frustum of a cone of revolution** let $b_1=\pi r_1^2$, and $b_2=\pi r_2^2$.

Then $$\sqrt{b_1b_1}=\sqrt{\pi r_1^2\pi r_2^2}=\pi r_1r_2.$$

Substituting these values in § 307,

$$v=\frac{1}{3}\pi h(r_1^2+r_2^2+r_1r_2).$$

EXERCISES

1. The bases of a frustum of a pyramid are regular hexagons whose sides are 8 in. and 4 in. respectively. The altitude of the frustum is 3 feet. Find the volume.

2. The radii of the bases of a frustum of a cone of revolution are 4 in. and 5 inches. The frustum is 12 in. high. Find the volume.

3. A conical heap of grain is 4 ft. high and has a circular base whose radius is 5 feet. How high must a bin be whose base is 4 ft. square to contain the grain?

4. Find the number of bushels of wheat contained in conical heap thrown into a corner of a bin, the highest point of the heap being 4 ft. and the radius of the circular base being 6.5 feet?

A bushel contains 2,150 cubic inches.

5. A cone is 12 in. high and the area of the base is 15 square inches. Find the volume.

6. The height of the frustum of a cone is 6 in. and the radii of the bases are 4 in. and 8 in. respectively. Find the volume.

7. What must be the depth of a pail that is 18 in. across the top and 10 in. across the bottom, in order that it may hold 5,280 cubic inches? ($\pi = \frac{22}{7}$.) (Yale.)

8. A pyramid is 6 in. high. The area of its base is 324 square inches. Find the volume of the frustum cut off by a plane 4 in. from the base.

FIG. 193

9. Find the volume of a grain tank, Fig. 193, 10 ft. high and 9 ft. in diameter, the height of the roof being 3 feet.

Volume of a Sphere

309. Theorem: *The volume of a sphere of radius r is* $\frac{4}{3}\pi r^3$.

Let ACB, Fig. 194, be a hemisphere, and let DF be a right cylinder, whose circular base, DE, is equal to the

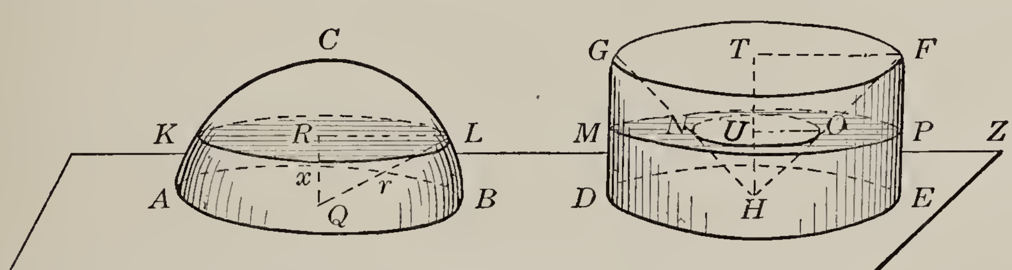


FIG. 194

circle AB and whose altitude is equal to the radius, r, of the sphere.

Suppose a cone $H-GF$ be cut from the cylinder leaving the solid, $GDEFHG$.

Pass a plane parallel to plane Z and at a distance x from Z, and let KL and $MNOP$ be the sections of ACB and $GDEFH$, respectively.

Show that $RL=\sqrt{r^2-x^2}$.

Show that $\frac{UO}{TF}=\frac{HU}{HT}$, or $\frac{UO}{r}=\frac{x}{r}$.

$\therefore \quad OU=x$.

Show that the area of $KL=\pi(r^2-x^2)$.

Show that the area of the circular ring $MNOP$

$$=\pi r^2-\pi x^2=\pi(r^2-x^2).$$

$\therefore$ the hemisphere ACB is equal to the solid $GDEFHG$, §302.

Since $\quad GDEFHG=(DF)-(H-GF)$

$$=\pi r^2\cdot r-\tfrac{1}{3}\pi r^2\cdot r=\tfrac{2}{3}\pi r^3,$$

it follows that the hemisphere $ACB=\frac{2}{3}\pi r^3$.

$\therefore$ the volume of the sphere is given by the formula

$$v=\tfrac{4}{3}\pi r^3.$$

EXERCISES

1. Find the weight of a cast-iron sphere 4 in. in diameter.

Cast iron weighs .26 lb. per cubic inch.

2. Find the volume of a sphere 4 in. in diameter.

3. Find the volume of metal in a spherical shell $\frac{1}{2}$ in. thick whose external diameter is 4 inches.

4. The area of a spherical surface is 6 square inches. Find the volume of the sphere.

5. A bar of metal of the form of a rectangular parallelopiped $12\times8\times4$ in. is to be melted and cast into a spherical ball. What is the radius of the ball?

No allowance is to be made for waste.

6. Prove that the volumes of two spheres are to each other as the cubes of the radii.

7. Regarding the earth and the sun as spheres of radii 4,000 mi. and 860,000 mi., respectively, compare their volumes.

8. A rifle shell has the shape of a cylinder surmounted by a hemispherical cap. The total length of the shell is four times its diameter. Compare the surfaces and also the volumes of the cylindrical and the spherical portions. (Sheffield.)

9. Find the volume and surface of a sphere inscribed in a cube whose diagonal is $6\sqrt{3}$. (Yale.)

10. A sphere is inscribed in a cube. Find the ratio of the radius of the sphere to the edge of the cube.

11. What percentage of the volume of a sphere is contained in the inscribed cube? (Harvard.)

12. A regular octaedron has an edge a. Find the volume of the inscribed sphere. (Harvard.)

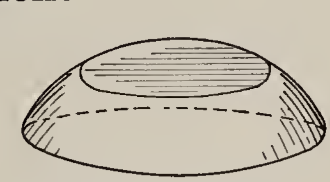
13. In a semicircle of radius a is inscribed a right triangle one of whose acute angles is 30°, the hypotenuse of the triangle being the diameter of the circle. The figure is revolved about the diameter as an axis. Find the ratio of the volumes generated by the triangle and the semicircle. (Yale.)

14. The inside of a glass is in the form of a cone whose vertical angle is 60°, and whose base is 2 in. across. The glass is filled with water and the largest sphere that can be immersed is placed in the glass. How much water remains in the glass? (Yale.)

15. A hemisphere and a right circular cone have the same base, and the areas of their curved surfaces are equal. Find the ratio of their volumes. (Harvard.)

Volume of a Spherical Segment

310. Spherical segment. The portion of a sphere included between two parallel planes intersecting a sphere is a spherical segment, Fig. 195.



Spherical Segment of two bases

Fig. 195

The perpendicular between the planes is the *altitude*, the sections of the sphere made by the planes are the *bases* of the segment.

If one of the planes is tangent to the sphere the segment has only one base, Fig. 196.

Spherical Segment of one base

FIG. 196

311. Theorem: *The volume of a spherical segment of one base is given by the formula*

$$V = \frac{1}{3}h^2\pi(3r - h),$$

where r is the radius of the sphere and h the altitude of the segment.

Proof: According to §§ 302, 309, the segment KCL, Fig. 197, is equal to the solid $MPFONG$, which is the

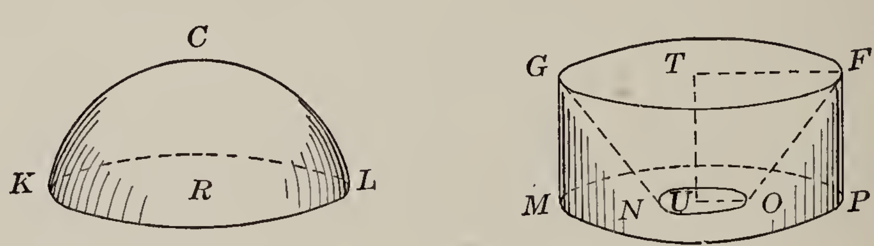


FIG. 197

difference between the cylinder MF and the frustum of a cone, $GNOF$.

The radii of the bases of the frustum are $TF = r$, and $UO = x = r - h$.

$\therefore \quad MF = \pi r^2 h$

and $\quad GNOF = \frac{1}{3}\pi h[r^2 + (r-h)^2 + r(r-h)]$

$\qquad = \frac{1}{3}\pi h(3r^2 - 3rh + h^2)$

$\therefore \quad KCL = MF - GNOF = \pi r^2 h - \frac{1}{3}\pi h(3r^2 - 3rh + h^2)$

$\qquad = \frac{1}{3}\pi h(3r^2 - 3r^2 + 3rh - h^2)$

$\qquad = \frac{1}{3}\pi h(3rh - h^2)$

$\therefore \quad V = \frac{1}{3}\pi h^2(3r - h)$

312. Theorem: *The volume of a spherical segment of two bases is given by the formula*

$$V=\frac{h}{2}(\pi r_1^2+\pi r_2^2)+\frac{\pi h^3}{6}.$$

Proof: The volume of a segment of two bases is equal to the difference of two segments having one base.

Denoting the altitudes of the segment of two bases by h and the altitudes of the segments of one base by h_1 and h_2, respectively, we have $h=h_1-h_2$.

$$\begin{aligned}\therefore\quad v&=\tfrac{1}{3}\pi h_1^2(3r-h_1)-\tfrac{1}{3}\pi h_2^2(3r-h_2)\\&=\pi r h_1^2-\tfrac{1}{3}\pi h_1^3-\pi r h_2^2+\tfrac{1}{3}\pi h_2^3.\\&=\pi r(h_1^2-h_2^2)-\tfrac{1}{3}\pi(h_1^3-h_2^3)\\&=\pi r(h_1-h_2)(h_1+h_2)-\tfrac{1}{3}\pi(h_1-h_2)(h_1^2+h_1h_2+h_2^2)\\&=\pi r(h_1-h_2)[r(h_1+h_2)-\tfrac{1}{3}(h_1^2+h_1h_2+h_2^2)]\\&=\pi h[rh_1+rh_2-\tfrac{1}{3}(h_1^2-2h_1h_2+h_2^2+3h_1h_2)]\\&=\pi h[rh_1+rh_2-\tfrac{1}{3}(h^2+3h_1h_2)]\\&=\pi h\left(rh_1+rh_2-\frac{h^2}{3}-h_1h_2\right)\end{aligned}$$

Show that $\dfrac{h_2}{r_2}=\dfrac{r_2}{2r-h_2}$ and that $\dfrac{h_1}{r_1}=\dfrac{r_1}{2r-h_1}$,

$$\therefore\quad 2rh_2-h_2^2=r_2^2$$

and

$$2rh_1-h_1^2=r_1^2.$$

Adding, $2rh_1+2rh_2-(h_1^2+h_2^2)=r_1^2+r_2^2$

$$\therefore\quad rh_1+rh_2=\frac{r_1^2+r_2^2}{2}+\frac{h_1^2+h_2^2}{2}$$

$$\begin{aligned}\therefore\quad v&=\pi h\left(\frac{r_1^2+r_2^2}{2}+\frac{h_1^2+h_2^2}{2}-\frac{h^2}{3}-h_1h_2\right)\\&=\pi h\left(\frac{r_1^2+r_2^2}{2}+\frac{h^2+2h_1h_2}{2}-\frac{h^2}{3}-\frac{2h_1h_2}{2}\right)\\&=\frac{\pi h}{2}\left(r_1^2+r_2^2+h^2-\frac{h^2}{3}\right)\end{aligned}$$

$$\therefore\quad V=\frac{h}{2}(\pi r_1^2+\pi r_2^2)+\frac{\pi h^3}{6}$$

313. Spherical cone. **A spherical cone,** Fig. 198, *is* generated by revolving a circular sector, ABC, Fig. 199, about its bounding radius, BC, as an axis.

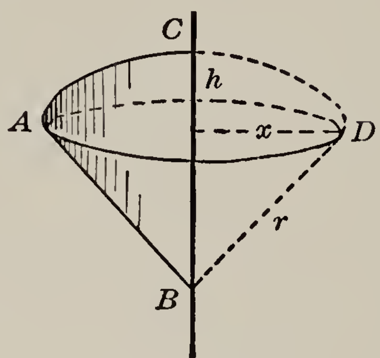


Fig. 198

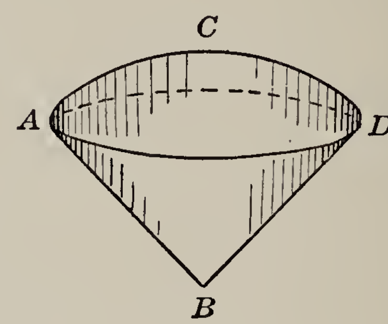


Fig. 199

314. Theorem: *The volume of a spherical cone is given by the formula.*

$$v=\frac{2}{3}\pi r^2 h.$$

Proof: $ABDC = ABD + ACD$

$$\begin{aligned} &= \tfrac{1}{3}\pi x^2(r-h)+\tfrac{1}{3}\pi h^2(3r-h) \\ &= \tfrac{1}{3}\pi[r^2-(r-h)^2](r-h)+\tfrac{1}{3}\pi h^2(3r-h) \\ &= \tfrac{1}{3}\pi(2rh-h^2)(r-h)+\tfrac{1}{3}\pi h^2(3r-h) \\ &= \tfrac{1}{3}\pi h(2r^2-hr-2rh+h^2+3rh-h^2) \\ &= \tfrac{1}{3}\pi h\cdot 2r^2 \end{aligned}$$

$$\therefore \quad v=\frac{2}{3}\pi r^2 h.$$

315. Spherical sector. The portion of a sphere generated by revolving a circular sector ABC, Fig. 200, about a diameter of its circle is a **spherical sector,** Fig. 201.

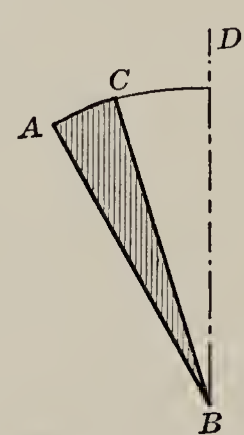


Fig. 200

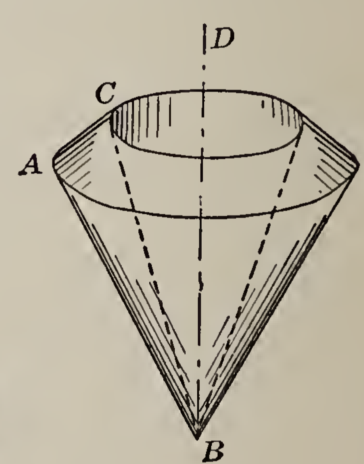


Fig. 201

316. Theorem: *The volume of a spherical sector is given by the formula*

$$v=\frac{2}{3}\pi r^2 h.$$

Proof: A spherical sector is the difference between two spherical cones. Denote their volumes by v_1 and v_2 respectively.

Then
$$v_1 = \tfrac{2}{3}\pi r^2 h_1$$
$$v_2 = \tfrac{2}{3}\pi r^2 h_2$$
$$\therefore \quad v_1 - v_2 = \tfrac{2}{3}\pi r^2 (h_1 - h_2)$$

or
$$\boldsymbol{v = \tfrac{2}{3}\pi r^2 h.}$$

EXERCISES

1. The distance of a plane from the center of a sphere is one-third the radius of the sphere. Find the ratio of the volumes of the two solids into which the sphere is divided by this plane. (Harvard.)

2. In a certain sphere there are as many square feet in the surface as there are cubic feet in the volume. Find the radius and determine the area of the segment of this spherical surface cut off by a plane perpendicular to the radius at its middle point.

‡**3.** How large a hole must be bored through a sphere 6 in. in diameter to remove one-half of the sphere?

The part cut from the sphere consists of a cylinder, C, and two spherical segments, S.

Show that $2S = \dfrac{2\pi r^3}{3}(2 + \cos x - 2\cos^2 x - \cos^3 x)$

and that $\quad C = 2\pi r^3(\cos x - \cos^3 x).$

‡**4.** The diameter of a sphere is 10 inches. If a cylindrical hole of 5 in. in diameter is bored through the sphere, what is the volume of the remaining solid? It is assumed that the center of the sphere lies on the axis of the cylinder.

5. The curved surface of a spherical segment of one base is 25π and the altitude is 3. Find the volume.

Summary

317. The chapter has taught the meaning of the following terms:

unit of volume, volume
inscribed prism, pyramid, and frustum of a pyramid
circumscribed prism, pyramid, and frustum of a pyramid
tangent plane
spherical segment, cone, and sector

318. The following theorems have been studied:

1. *The plane passed through two diagonally opposite edges of a right parallelopiped divides the parallelopiped into two equal triangular right prisms.*

2. *An oblique prism is equal to a right prism whose base is equal to a right section of the oblique prism and whose altitude is equal to the lateral edge of the oblique prism.*

3. *The plane passed through two diagonally opposite edges of any parallelopiped divides the parallelopiped into two equal triangular prisms.*

4. *Prisms having equal bases and altitudes are equal.*

5. *The volumes of two similar cylinders of revolution are to each other as the cubes of the altitudes, or as the cubes of the radii of the bases.*

6. *If two pyramids have equal bases and equal altitudes, sections made by planes parallel to the bases and at equal distances from the vertices are equal.*

7. *If two triangular pyramids have equal bases and altitudes they are equal.*

8. *If two solids lie between two given parallel planes, having their bases in these planes, and if the sections made by any plane parallel to the given planes are equal, then the volumes of the solids are equal.*

319. The following is a summary of the formulas in this chapter:

Rectangular parallelopiped	$v=a\times b\times c$
	$v=b\times h$ (b=base)
Cube	$v=e^3$
Triangular right prism	$v=b\times h$
Right parallelopiped	$v=b\times h$
Oblique parallelopiped	$v=b\times h$
Triangular prism	$v=b\times h$
Prism	$v=b\times h$
Cylinder	$v=b\times h$
Cylinder of revolution	$v=\pi r^2h$
Pyramid	$v=\frac{1}{3}b\times h$
Frustum of a pyramid	$v=\frac{1}{3}h(b_1+b_2+\sqrt{b_1b_2})$
Cone	$v=\frac{1}{3}h\times b$
Frustum of a cone of revolution	$v=\frac{1}{3}\pi h(r_1^2+r_2^2+r_1r_2)$
Sphere	$v=\frac{4}{3}\pi r^3$
Spherical segment of one base	$v=\frac{1}{3}\pi h^2(3r-h)$
Spherical segment of two bases	$v=\frac{h}{2}(\pi r_1^2+\pi r_2^2)+\frac{\pi h^3}{6}$
Spherical cone	$v=\frac{2}{3}\pi r^2h$
Spherical sector	$v=\frac{2}{3}\pi r^2h$

CHAPTER XIV

POLYEDRAL ANGLES. TETRAEDRONS. SPHERICAL POLYGONS

Polyedral Angles

320. Polyedral angle. If a line, AB, Fig. 202, moves with one endpoint fixed at A and always touching a convex polygon, $CDEFG$, whose plane does not contain A, it generates a convex **polyedral angle.**

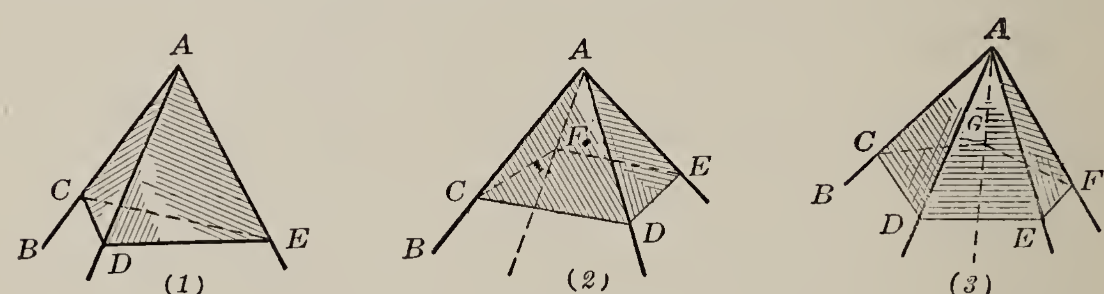


Fig. 202

The fixed point A is the **vertex,** the bounding planes CAD, DAE, etc., are the **faces,** the lines AC, AD, etc., are the **edges,** $\measuredangle CAD$, DAE, etc., are the **face angles** of the polyedral angle.

321. Triedral angle. A polyedral angle having three faces, as (1) Fig. 202, is a **triedral angle.**

Point out several triedral angles in the classroom.

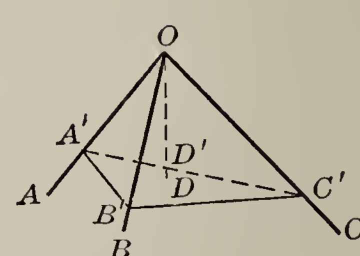


Fig. 203

322. Theorem: *The sum of two face angles of a triedral angle is greater than the third.*

Given the triedral angle $O\text{–}ABC$, Fig. 203.

To prove that $\angle AOB + \angle BOC > \angle AOC$.

Proof: The theorem is easily proved for a triedral angle having *equal* face angles.

Assume that the face angles are not equal and that $\angle AOC$ is the greatest face angle.

In the plane AOC draw OD, making $\angle AOD = \angle AOB$.

Lay off $OD' = OB'$.

Pass a plane through B' and D', cutting the faces in lines $A'B'$, $B'C'$, and $C'A'$ respectively.

Prove that $\triangle A'OB' \cong \triangle A'OD'$.

$$\therefore A'B' = A'D'.$$

Show that $\underline{A'B' + B'C' > A'C'.}$

Subtracting, $B'C' > D'C'$.

$$\therefore \angle B'OC' > \angle D'OC'.$$

$$\therefore \angle A'OB' + \angle B'OC' > \angle A'OD' + \angle D'OC',$$

or $$\angle AOB + \angle BOC > \angle AOC.$$

EXERCISES

1. Show that the difference of two face angles of a triedral angle is less than the third.

2. Show that any face angle of a polyedral angle is less than the sum of the other face angles.

3. Show that the three planes bisecting a triedral angle intersect in a straight line.

323. Spherical polygon. Let the faces of the polyedral angle O–$ABCD$, Fig. 204, intersect the surface of a

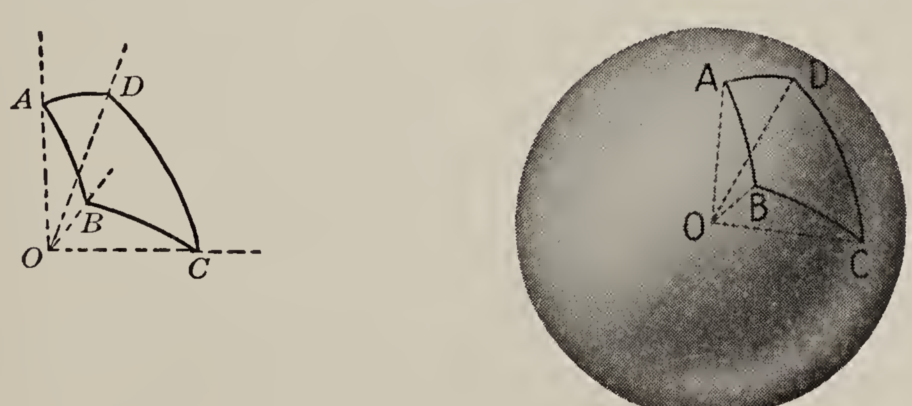


Fig. 204

sphere whose center is O in the great circle arcs AB, BC, CD, and DA. The figure $ABCD$ on the surface of the sphere is a *spherical polygon.*

Thus a **spherical polygon** is the section of a spherical surface made by a convex polyedral angle whose vertex is at the center of the sphere. To every polyedral angle at the center of the sphere corresponds a spherical polygon.

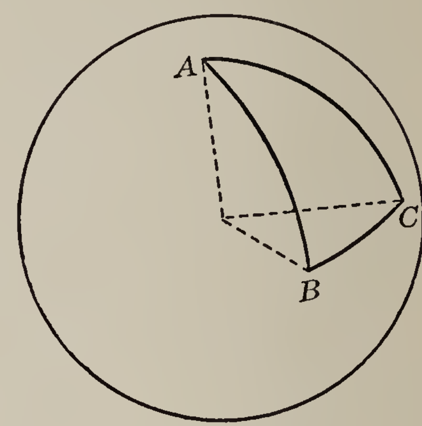


FIG. 205

A spherical polygon of *three* sides is a **spherical triangle,** Fig. 205.*

The bounding arcs, AB, BC, etc., are the **sides** of the spherical polygon. The points of intersection of the sides are the **vertices** of the polygon.

EXERCISES

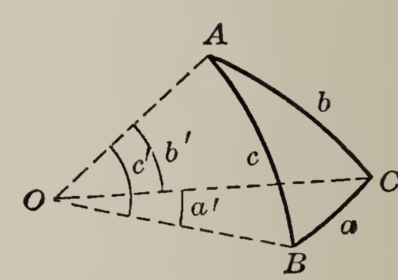


FIG. 206

1. The sides of a spherical polygon are usually measured in degrees. Show that *the sides of a spherical polygon have the same measure as the face angles of the corresponding polyedral angle at the center of the sphere,* Fig. 206.

* The properties of spherical triangles are applied in the solution of problems in astronomy, navigation, and geography. In fact, spherical geometry was first developed by astronomers. The following are some of the interesting applications:

1. To determine the position of an observer on the surface of the earth, i.e., his latitude and longitude.
2. To find the distance between two places and the bearing of each from the other, when their latitudes and longitudes are known.
3. To determine the position of a star.
4. To determine the time of the day at a place on the surface of the earth.
5. To determine the course of a ship.

2. Two sides of a spherical triangle are 88° and 70°. What are the limits for the third side?

Exercise 2 indicates how some properties of spherical polygons may be inferred from a study of polyedral angles.

3. Show that *the sum of two sides of a spherical triangle is greater than the third side,* Fig. 206.

4. *The shortest line that can be drawn between two given points on the surface of a sphere is the minor arc of the great circle which passes through the two points.* Prove.

Proof: 1. Let A and B, Fig. 207, be the two given points and let $\overset{\frown}{ACB}$ be the minor arc of a great circle joining A and B.

Let C be *any* point on $\overset{\frown}{AB}$. With A and B as centers and radii equal to AC and BC, respectively, draw two small circles meeting at C.

Let D be any point on circle A, *not* point C and draw the arcs of great circles $\overset{\frown}{AD}$ and $\overset{\frown}{DB}$.

Then $\overset{\frown}{AD}+\overset{\frown}{DB}>\overset{\frown}{AB}$, § 323, exercise 3.

But $\overset{\frown}{AD} \qquad =\overset{\frown}{AC}$,

$\therefore \qquad \overset{\frown}{DB}>\overset{\frown}{CB}$.

Thus D lies outside of circle B.

$\therefore$ circles A and B are tangent to each other at C.

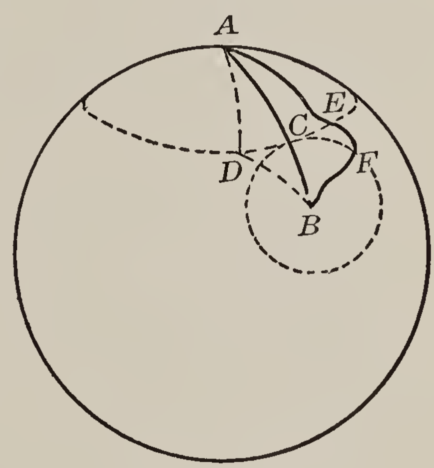


Fig. 207

2. Let $AEFB$ be any line joining A and B on the surface of the sphere and not passing through C. Then line $AEFB$ meets circles A and B in two distinct points, E and F. Why?

Whatever may be the form of AE, an equal line can be drawn from A to C; and whatever may be the form of BF, an equal line can be drawn from B to C. Why?

Hence it is possible to draw a line from A to B passing through C and equal to $AE+FB$.

Since $AE+FB<AE+EF+FB$, it is always possible to draw a line from A to B passing through C and shorter than $AEFB$.

Thus the shortest line from A to B passes through C.

Since C is *any* point on $\overset{\frown}{AB}$, the shortest line from A to B passes through every point of $\overset{\frown}{AB}$ and therefore *is the arc* AB.

324. Theorem: *The sum of the face angles of a convex polyedral angle is less than four right angles.*

Given the convex polyedral angle O–$ABCDE$, Fig. 208.

To prove that

$\angle AOB + \angle BOC + \angle COD +$ etc., < 4 R.A.

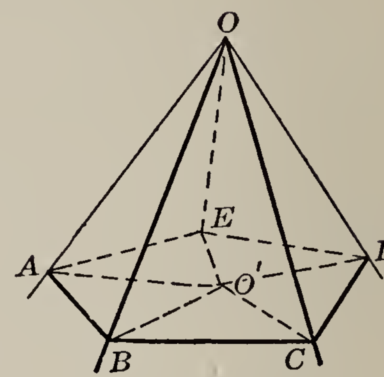


FIG. 208

Proof: Let $ABCDE$ be a section of the polyedral angle made by a plane cutting all the edges.

From any point O' within $ABCDE$ draw lines to the vertices A, B, C, etc.

In the triedral angle B–AOC,

$$\angle ABO + \angle OBC > \angle ABC, \text{ § 322.}$$

In the triedral angle C–BOD,

$$\angle BCO + \angle OCD > \angle BCD, \text{ etc.}$$

Adding,

$$\angle ABO + \angle OBC + \angle BCO + \angle OCD + \text{etc.,}$$
$$> \angle ABC + \angle BCD + \text{etc.,}$$

i.e., the sum of the *base* angles of the triangles with vertex O is *greater than* the sum of the *base* angles of the triangles with vertex O'.

But the sum of *all* angles of the triangles with vertex O is equal to the sum of *all* angles of the triangles with vertex O'.

$\therefore$ by subtracting unequals from equals we have the sum of the *face angles* at O less than the sum of the angles about O'.

In symbols this may be stated as follows:

$\angle AOB + \angle BOC + \angle COD +$ etc., $< \angle AO'B + \angle BO'C + \angle CO'D +$ etc.

Since

$\angle AO'B + \angle BO'C + \angle CO'D +$ etc. $= 4$ R.A.

$\therefore \angle AOB + \angle BOC + \angle COD +$ etc. < 4 R.A.

EXERCISE

Prove that *the sum of the sides of any convex spherical polygon* is less than 360°, Fig. 209.

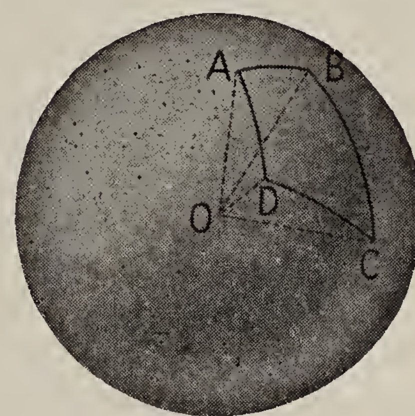


Fig. 209

325. Number of regular polyedrons. In § 245 *five* regular polyedrons were shown. The theorem in § 324 may be used to *prove* that there are *no other* kinds of convex regular polyedrons.

For the faces of a regular polyedron are all regular polygons such as equilateral triangles, squares, etc., and the sum of the face angles of any polyedral angle of the polyedron must be less than 360°. Why?

Show that *three, four,* or *five* equilateral triangles, but *not* six or more, may be placed so as to form a polyedral angle. Hence no polyedron can be formed with six or more equilateral triangles at the vertex. The *tetraedron* has three equilateral triangles at one vertex, the *octaedron* has four, and the *icosaedron* has five.

Show that *three* squares may be placed so as to form a polyedral angle, but *not* four or more. Hence no polyedron can be formed with four or more squares at a vertex. The *cube* has *three* squares at one vertex.

Show that *three* regular pentagons may be placed so as to form a polyedral angle, but *not* four or more. Hence the *dodecaedron* is the only polyedron whose faces are regular pentagons.

Show that it is impossible to form a regular polyedron having six or more regular polygons at one vertex.*

* The regular solids were studied so extensively by Plato and his school that they have received the name of "Platonic figures."

Tetraedrons

326. Theorem: *Two tetraedrons having a triedral angle of one equal to a triedral angle of the other are to each other as the products of the edges including the equal triedral angles.*

Given the tetraedrons $T\text{–}ABC$ and $T'\text{–}A'B'C'$, Fig. 210, with the triedral angle at T equal to the triedral

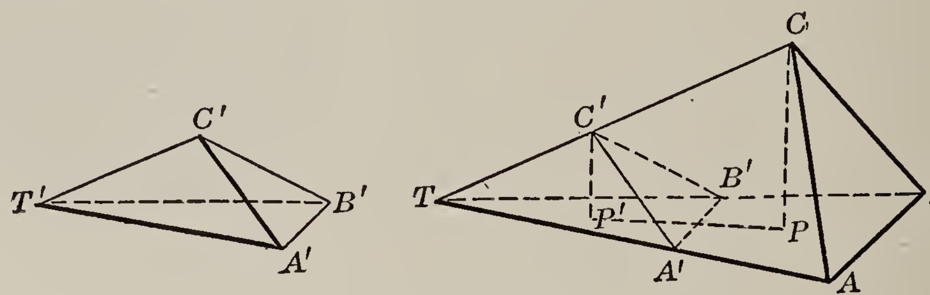


FIG. 210

angle at T' and having the volumes equal to V and V' respectively.

To prove that $\dfrac{V}{V'}=\dfrac{TA\times TB\times TC}{T'A'\times T'B'\times T'C'}$.

Proof: Place $T'\text{–}A'B'C'$ on $T\text{–}ABC$ making triedral angle T' coincide with triedral angle T.

Draw $C'P'$ and $CP\perp ATB$.

Then $$\frac{V}{V'}=\frac{\frac{1}{3}\times ABT\times CP}{\frac{1}{3}\times A'B'T\times C'P'}=\frac{ABT}{A'B'T}\times\frac{CP}{C'P'},\quad \text{§ 303.}$$

Since triangles ABT and $A'B'T$ have one angle equal,

$$\frac{ABT}{A'B'T}=\frac{TA\times TB}{TA'\times TB'},$$

Show that

$$\frac{CP}{C'P'}=\frac{TC}{TC'}.$$

By substitution,

$$\frac{V}{V'}=\frac{TA\times TB}{TA'\times TB'}\times\frac{TC}{TC'},$$

or $$\frac{V}{V'}=\frac{TA\times TB\times TC}{T'A'\times T'B'\times T'C'}.$$

327. Similar polyedrons. Two polyedrons are **similar** if their faces are similar each to each and similarly placed, and if the corresponding polyedral angles are equal.

328. Theorem: *Two similar tetraedrons are to each other as the cubes of the corresponding edges.*

$$\frac{V}{V'}=\frac{TA\times TB\times TC}{T'A'\times T'B'\times T'C'}, \quad \S\ 326.$$

Show that $\dfrac{TA}{T'A'}=\dfrac{TB}{T'B'}=\dfrac{TC}{T'C'}$.

$$\therefore \quad \frac{V}{V'}=\frac{TA}{T'A'}\times\frac{TA}{T'A'}\times\frac{TA}{T'A'}=\frac{\overline{TA}^3}{\overline{T'A'}^3}.$$

EXERCISE

A pyramid, the area of whose base is 36 sq. ft., contains $\frac{1}{36}$ of the volume of a similar pyramid whose altitude is 9 feet. Find the volume of each pyramid.

329. *To construct a sphere through four given points not all in the same plane.*

Given the four points A, B, C, and D, Fig. 211, not all in the same plane.

To construct a sphere passing through A, B, C, and D.

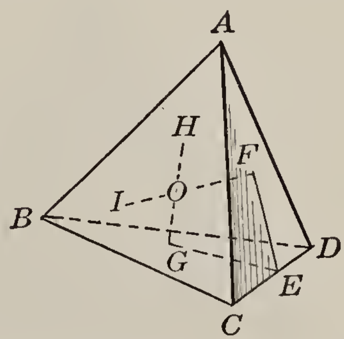


FIG. 211

Construction: Draw AB, AC, AD, BC, CD, and DB forming the tetraedron A–BCD.

Bisect CD at E.

Draw plane FEG perpendicular to CD at E, and intersecting planes CAD and CBD in lines EF and EG respectively.

Show that EF passes through the center F of the circle circumscribed about $\triangle CAD$, and that EG passes through the center G of the circle circumscribed about $\triangle CBD$.

Draw $FI \perp$ plane CAD and $GH \perp$ plane CBD.

Since $CD \perp$ plane FEG, it follows that planes CAD and CBD are perpendicular to plane FEG. Why?

Show that FI and GH lie in plane FEG, § 551.

Show that FI and GH are not parallel.

Denoting the point of intersection of FI and GH by O, show that O is equidistant from A, B, C, and D.

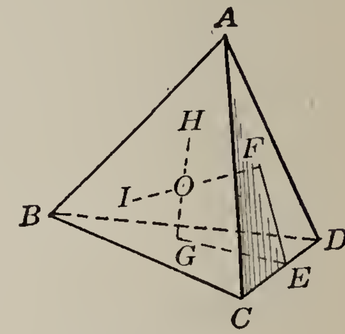


Fig. 211

Therefore a sphere with O as center and radius OB passes through A, B, C, and D.

330. *To inscribe a sphere in a given tetraedron.*

Given the tetraedron $A\text{–}BCD$, Fig. 212.

To construct a sphere tangent to all faces of $A\text{–}BCD$.

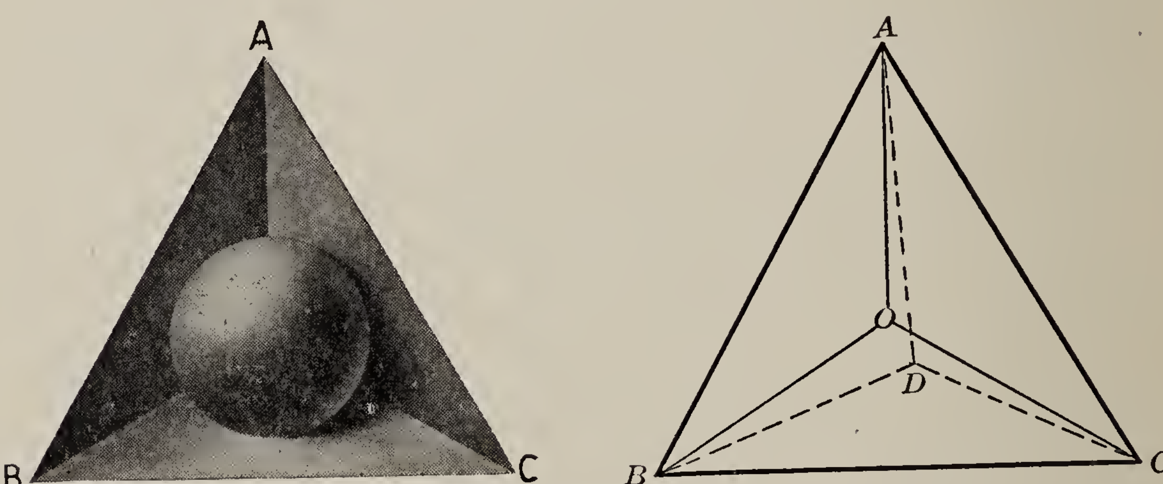


Fig. 212

Construction: Draw plane BOD bisecting the diedral angle BD.

Draw plane BOC bisecting diedral angle BC.

These planes must meet in a line, as BO, since they have point B in common.

Draw plane COA bisecting the diedral angle AC. This plane will meet the line of intersection of planes BOD and BOC in point O.

Show that O is equidistant from the four faces of the tetraedron A–BCD.

Hence a sphere with O as center and radius equal to the perpendicular from O to one of the faces will be tangent to all faces.

‡ **331.** *To determine the diameter of a given material sphere.*

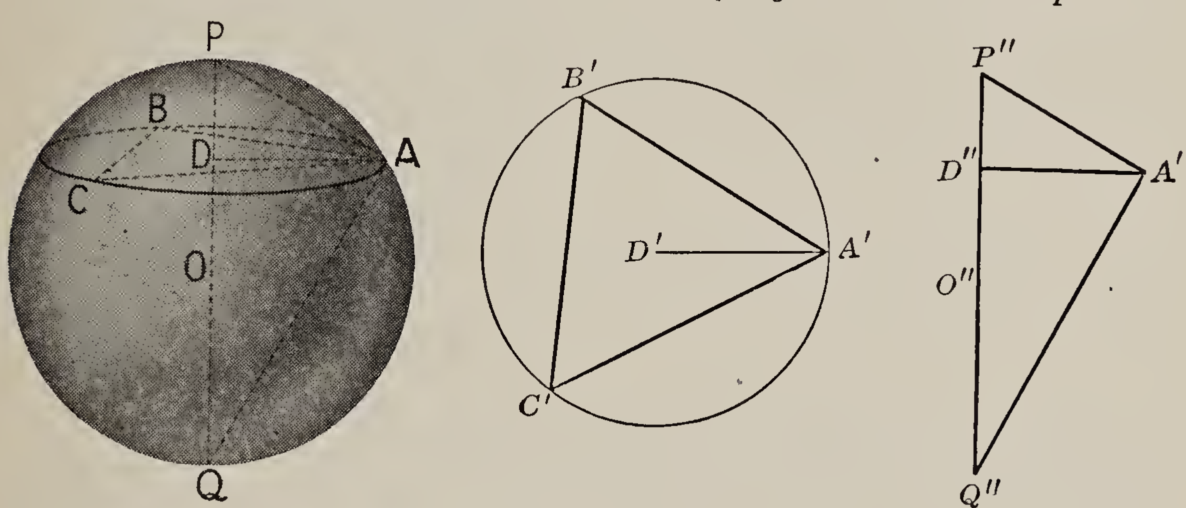


FIG. 213

Given a material sphere, O, Fig. 213.

To find its diameter.

Construction: With P, a point on the surface of the sphere, as a pole, describe the circle ABC.

Let A, B, and C be three points on this circle.

Construct triangle $A'B'C'$ congruent to the triangle ABC.

Circumscribe a circle about $\triangle A'B'C'$, and let D' be the center of this circle.

Draw $D''A''$ equal to the radius $D'A'$.

Through D'' draw a line $P''Q''$ perpendicular to $D''A''$.

From A'' lay off $A''P''$ equal to AP.

At A'' erect $A''Q''$ perpendicular to $A''P''$.

Then $P''Q''$ is the required diameter of the given sphere.

The proof is left to the student.

Spherical Angles

332. Spherical angle. Two intersecting curves, C and C', Fig. 214, are said to *form an angle*. The *angle formed by two intersecting curves* is the angle made by the tangents to the curves at the common point, as $\angle TOT'$.

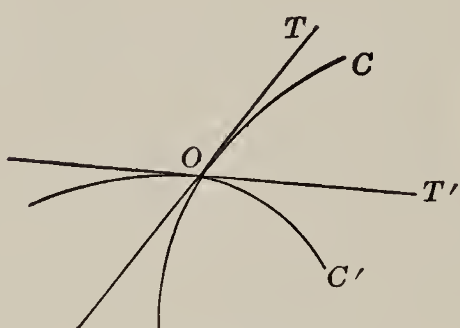


FIG. 214

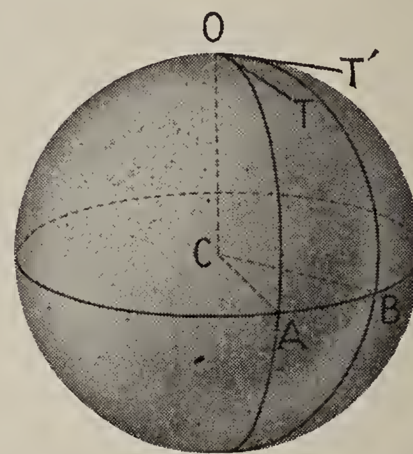


FIG. 215

The angle formed by two intersecting arcs of great circles is a **spherical angle,** as TOT', Fig. 215. The point of intersection, O, is the *vertex* and the arcs OA and OB are the *sides* of the spherical angle.

333. Measure of a spherical angle. Draw AB, Fig. 215, an arc of a great circle with O as a pole and terminated by the sides of the spherical angle AOB.

Draw the radii CO, CA, and CB.

Show that OC is perpendicular to OT and CA,

$$\therefore \quad OT \parallel CA.$$

Similarly, $$OT' \parallel CB.$$

$$\therefore \quad \angle TOT' = \angle ACB.$$

But $\angle ACB$ has the same measure as arc AB.

$\therefore$ $\angle TOT'$, or spherical angle AOB, is measured by arc AB.

This proves the following theorem:

A spherical angle is measured by the arc of a great circle having the vertex as pole, and included between the sides, produced if necessary.

EXERCISES

1. Prove that *a spherical angle is equal to the diedral angle formed by the planes of the sides.*

2. The angles formed by the sides of a spherical triangle are respectively equal to the diedral angles of the corresponding triedral angle. Prove.

334. Right spherical angle. When the planes of the sides of an angle are perpendicular to each other *a right spherical angle* is formed.

335. Classification of spherical triangles. The terms **isosceles, equilateral,** and **scalene** have the same meaning for spherical triangles as for plane triangles. A spherical triangle is *right, birectangular,* or *trirectangular*

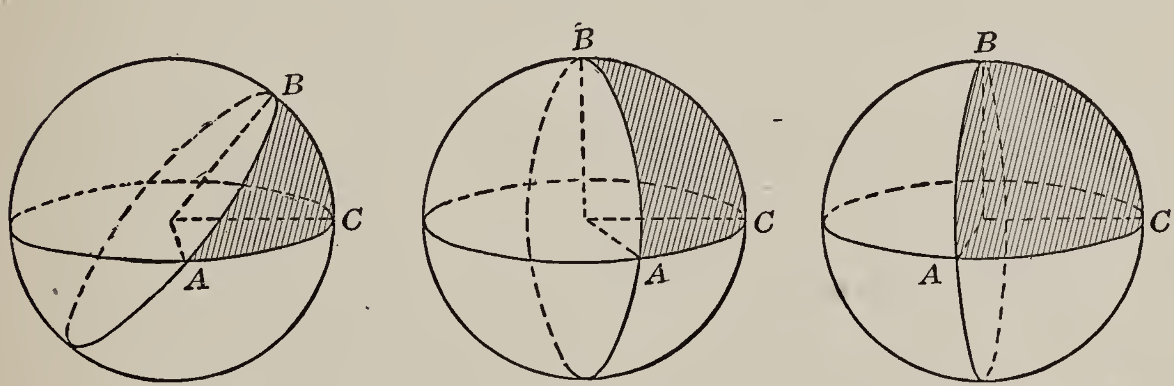


FIG. 216

according as it has one, two, or three right angles, Fig. 216.

Polar Spherical Triangles

336. Polar triangles. Let $\triangle ABC$, Fig. 217, be a given triangle. Draw three great circle arcs as $A'B'$ $B'C'$, and $C'A'$, having as poles C, A, and B, respectively. Any two of these circle arcs, if far enough extended, have two points of intersection. Let C' be that point of intersection of arcs $A'C'$ and $B'C'$ which is nearest to C, let B' be the point of intersection of arcs $A'B'$ and $C'B'$ which is nearest to B, and let A' be the point of intersection of arcs $B'A'$ and $C'A'$ which is nearest to A. Then $\triangle A'B'C'$ is the **polar triangle** of $\triangle ABC$.

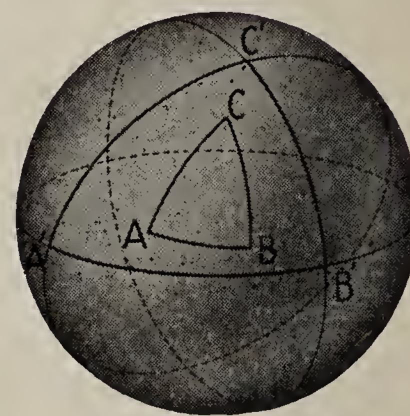


Fig. 217

337. Theorem: *If a spherical triangle is the polar triangle of another, then the second is the polar triangle of the first.*

Given $\triangle ABC$, Fig. 218 and $\triangle A'B'C'$, the polar of $\triangle ABC$.

To prove that $\triangle ABC$ is the polar of $\triangle A'B'C'$.

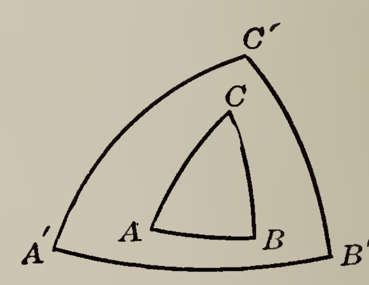


Fig. 218

Proof: Since A is the pole of $B'C'$, it follows that B' is a quadrant's distance from A.

Since C is the pole of $B'A'$, it follows that B' is also a quadrant's distance from C.

$\therefore$ B' is the pole of AC, § 563.

Similarly, prove that A' is the pole of BC, and C' is the pole of AB.

$\therefore$ $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

EXERCISES

1. Make a sketch of the polar triangle of a birectangular triangle and show that it is also birectangular.

2. Show that the polar triangle of a given trirectangular triangle is identical with the given triangle.

338. Theorem: *In two polar spherical triangles, each angle of the one is the supplement of that side of the other of which it is the pole.*

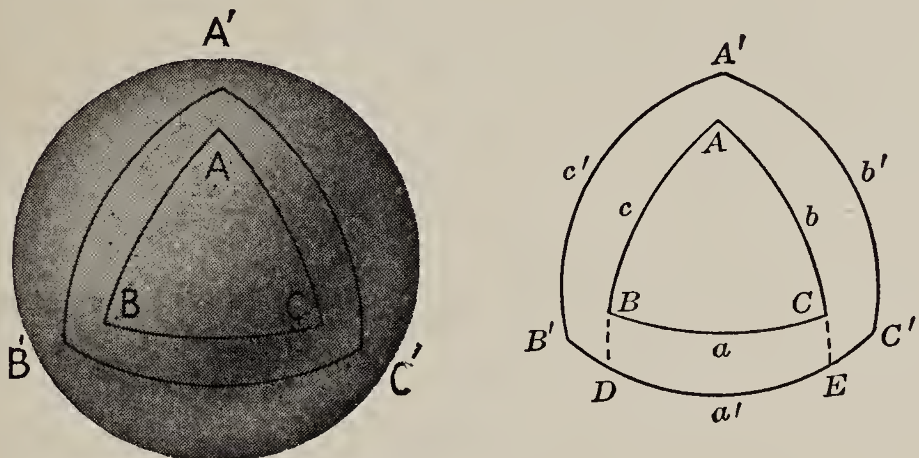


FIG. 219

Given the polar triangles $A'B'C'$ and ABC, Fig. 219, having the sides equal to a', b', c' and a, b, c respectively.

To prove that $A+a'=180°$, $B+b'=180°$, $C+c'=180$,

$$A'+a=180°, B'+b=180°, C'+c=180°.$$

Proof: Let the sides of $\angle A$, produced if necessary, intersect the side $B'C'$ in points D and E respectively.

Since $\angle A$ is measured by arc DE, § 333, $A=\overset{\frown}{DE}$.

Since $\overset{\frown}{B'E}=90°=\overset{\frown}{DC'}$, $\overset{\frown}{B'E}+\overset{\frown}{DC'}=180°$.

$\overset{\frown}{B'E}+(\overset{\frown}{DE}+\overset{\frown}{EC'})=180°$. Why?

$\overset{\frown}{DE}+(\overset{\frown}{B'E}+\overset{\frown}{EC'})=180°$. Why?

$\overset{\frown}{DE}+\overset{\frown}{B'C'}=180°$. Why?

$\therefore\ A+a'=180°$. Why?

Similarly, $B+b'=180°$, $C+c'=180°$, etc.

EXERCISES

1. Find the sides of the polar triangle of a triangle whose angles are 75°, 85°, and 88° respectively.

2. The angles of a spherical triangle are 88°, 125°, and 96° respectively. Find the sides of the polar triangle.

339. Theorem: *The sum of the angles of a spherical triangle is less than six and greater than two right angles.*

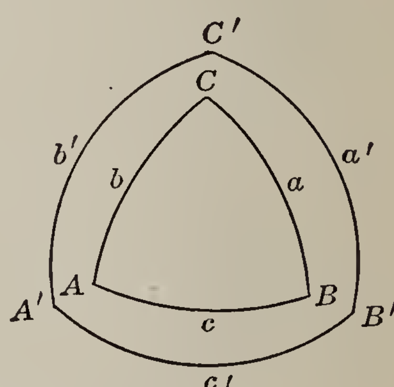


FIG. 220

Given the spherical triangle ABC, Fig. 220.

To prove that $A+B+C<540°$; $A+B+C>180°$.

Proof: 1. Let $\triangle A'B'C'$ be the polar triangle of $\triangle ABC$.

Then
$$A+a'=180°$$
$$B+b'=180°$$
$$C+c'=180°$$

Adding,

$$A+B+C+a'+b'+c'=540°$$
$$\therefore \quad A+B+C=540°-(a'+b'+c')$$
or
$$A+B+C<540°$$

2. $A+B+C+a'+b'+c'=540°$

$$360°>a'+b'+c' \qquad \text{(See p. 307.)}$$

Adding,

$$A+B+C+a'+b'+c'+360°>540°+a'+b'+c'$$
$$\therefore \quad A+B+C>180°$$

340. Spherical excess. The amount by which the sum of the angles of a spherical triangle *exceeds* 180° is called the **spherical excess.**

EXERCISES

1. Show that the spherical excess of a triangle is equal to $A^\circ + B^\circ + C^\circ - 180^\circ$.

2. The angles of a spherical triangle are 100°, 65°, and 190°. Find the spherical excess.

Symmetry and Congruence

341. Congruent polyedral angles. Two polyedral angles are **congruent** if they can be made to coincide.

It follows that the face angles and diedral angles of one of two congruent polyedral angles are equal respectively to those of the other. However, it does not follow that two polyedral angles are *congruent* if the corresponding face angles and diedral angles are equal.

For example, triedral angles $O\text{–}ABC$, $O'\text{–}A'B'C'$ and $O''\text{–}A''B''C''$, Fig. 221, have the corresponding face angles and diedral angles equal.

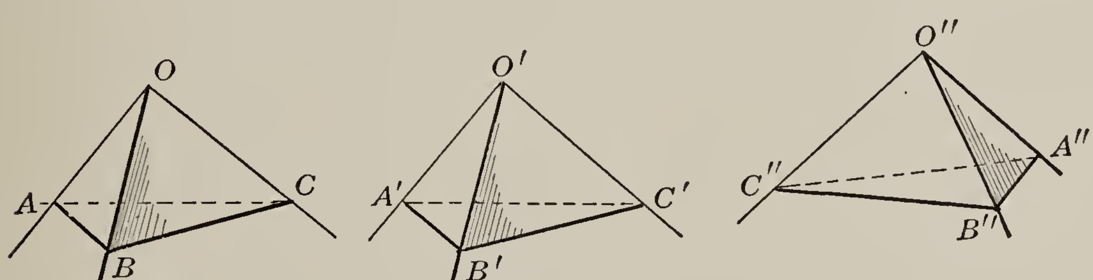


FIG. 221

$O\text{–}ABC$ and $O'\text{–}A'B'C'$ are congruent, but $O\text{–}ABC$ and $O''\text{–}A''B''C''$ are *not* congruent.

342. Symmetrical polyedral angles. Two polyedral angles are **symmetrical** if the face angles and diedral angles of one are equal respectively to the face angles and diedral angles of the other, but arranged in *opposite order.*

343. Congruent and symmetrical spherical polygons. If the sides and angles of one spherical polygon are equal respectively to those of another, the polygons are **congruent,** provided the parts are arranged in the *same* order, and **symmetrical,** provided the parts are arranged in *opposite order.*

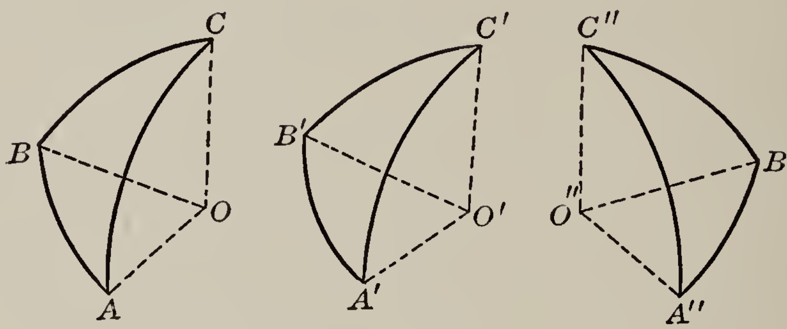


FIG. 222

Thus, in Fig. 222, $ABC \cong \triangle A'B'C'$ and both triangles are symmetrical to $\triangle A''B''C''$.

344. Theorem: *If two triedral angles have the three face angles of one equal respectively to the three face angles of the other, the corresponding diedral angles are equal.*

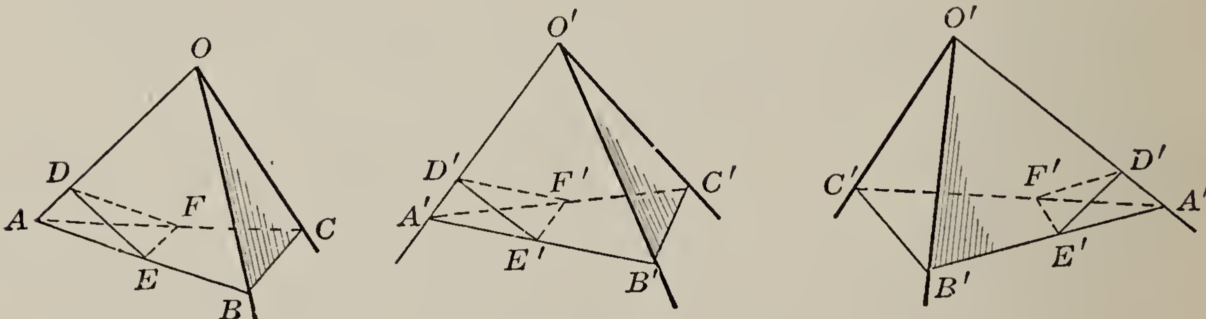


FIG. 223

Given the triedral angles O–ABC and O'–$A'B'C'$, Fig. 223, having $\angle AOB = \angle A'O'B'$, $\angle BOC = \angle B'O'C'$, $\angle COA = \angle C'O'A'$.

To prove that diedral angle AO = diedral angle $A'O'$, $BO = B'O'$, and $CO = C'O'$.

Proof: Lay off $OA = OB = OC = O'A' = O'B' = O'C'$.

Draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Prove that $\triangle AOB \cong \triangle A'O'B'$; $\triangle BOC \cong \triangle B'O'C'$; $\triangle COA \cong \triangle C'O'A'$.

Prove that $\triangle ABC \cong \triangle A'B'C'$.

Take $OD = O'D'$.

Draw DE and DF perpendicular to AO and in faces AOB and AOC respectively.

Similarly, draw $D'E'$ and $D'F'$.

Prove that $\triangle EDA \cong E'D'A'$; $\triangle FDA \cong \triangle F'D'A'$.

Prove that $\triangle EAF \cong \triangle E'A'F'$.

Prove that $\triangle EDF \cong \triangle E'D'F'$.

$\therefore \angle EDF = \angle E'D'F'$.

$\therefore$ Diedral $\angle AO =$ diedral $\angle A'O'$, having equal plane angles.

Similarly, diedral $\angle BO =$ diedral $\angle B'O'$.

diedral $\angle CO =$ diedral $\angle C'O'$.

EXERCISES

Prove the following:

1. *Two triedral angles are congruent if the face angles of one are equal respectively to the face angles of the other, arranged in the same order.*

2. *Two triedral angles are symmetrical if the face angles of one are equal respectively to the face angles of the other, arranged in the reverse order.*

3. *If two spherical triangles on the same sphere or on equal spheres have three sides of one equal respectively to three sides of the other—*

1. *They are congruent if the equal parts are arranged in the same order.*

2. *They are symmetrical if the equal parts are arranged in the reverse order.*

Show that the triangles being mutually equilateral are also mutually equiangular, and therefore either congruent or symmetrical.

4. *Two spherical triangles on the same or equal spheres are congruent—*

1. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other, arranged in the same order.*

2. *If two angles and the included side of one are equal respectively to two angles and the included side of the other, arranged in the same order.*

Show that $\triangle ABC$, Fig. 224, can be made to coincide with $\triangle A'B'C'$.

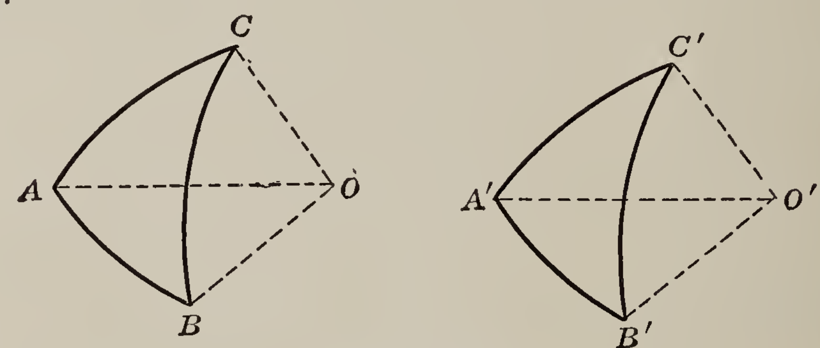


Fig. 224

5. *Two spherical triangles on the same or equal spheres are symmetrical—*

1. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other, arranged in the reverse order.*

2. *If two angles and the included side of one are equal respectively to two angles and the included side of the other, arranged in the reverse order.*

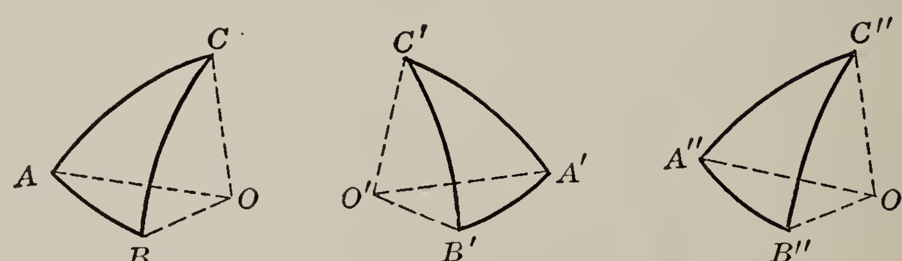


Fig. 225

Let $\triangle ABC$ and $A'B'C'$, Fig. 225, be the given triangles.
Draw $\triangle A''B''C''$ symmetrical to $\triangle A'B'C'$.

Then $\triangle A'B'C'$ and $A''B''C''$ are mutually equilateral and mutually equiangular.

Prove that $\triangle ABC \cong \triangle A''B''C''$.

$\therefore$ $\triangle ABC$ is symmetrical to $\triangle A'B'C'$.

6. State and prove theorems on triedral angles corresponding to the theorems in exercises 3, 4, and 5.

7. *If two spherical triangles on the same or equal spheres are mutually equiangular, they are mutually equilateral and congruent, if the equal parts are arranged in the same order. If the equal parts are arranged in the reverse order, they are symmetrical.*

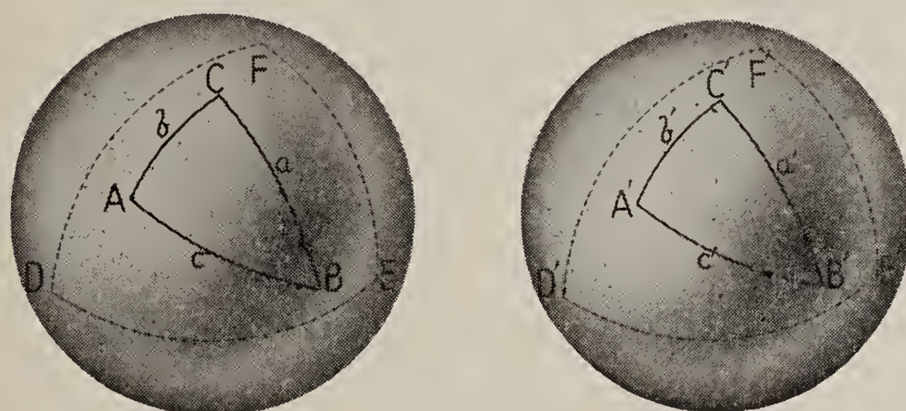


Fig. 226

Given the spherical triangles ABC and $A'B'C'$, Fig. 226, having $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

To prove that $a = a'$, $b = b'$, $c = c'$, and that $\triangle ABC$ and $A'B'C'$ are either congruent or symmetrical.

Proof: Construct the polar triangles DEF and $D'E'F'$.

Show that $\triangle DEF$ and $D'E'F'$ are mutually equilateral, § 338.

Show that $\triangle DEF$ and $D'E'F'$ are mutually equiangular, exercise 3.

Show that $\triangle ABC$ and $A'B'C'$ are mutually equilateral, § 338.

$\therefore$ ABC and $A'B'C'$ are either congruent or symmetrical, § 343.

345. Theorem: *The angles opposite the equal sides of an isosceles spherical triangle are equal.*

Draw the arc of a great circle, CD, Fig. 227, bisecting the side AB.

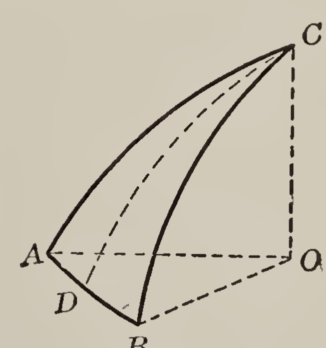


Fig. 227

Then $\triangle ADC$ and BDC are mutually equilateral and therefore symmetrical,

$$\therefore \quad \angle A = \angle B$$

EXERCISES

1. State and prove the theorem on triedral angles corresponding to § 345.

2. *Two symmetrical isosceles triangles are congruent.* Prove.

346. Theorem: *If two angles of a spherical triangle are equal, the sides opposite are equal.*

Construct the polar triangle of $\triangle ABC$, Fig. 228.

Since $\angle A = \angle B$,

it follows that $\overset{\frown}{B'C'} = \overset{\frown}{A'C'}$, § 338.

$\therefore \quad \angle B' = \angle A'$, § 345.

$\therefore \quad \overset{\frown}{AC} = \overset{\frown}{BC}$, § 338.

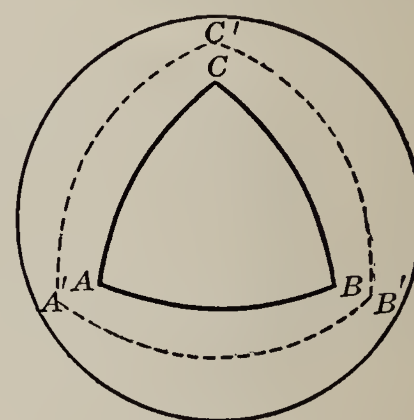


Fig. 228

347. Theorem: *If two angles of a spherical triangle are unequal, the sides opposite are unequal and the greater side lies opposite the greater angle.*

Draw AD, Fig. 229, an arc of a great circle making

$$\angle DAB = \angle DBA.$$

Then $\overset{\frown}{AC} < \overset{\frown}{CD} + \overset{\frown}{DA}$,

or $\overset{\frown}{AC} < \overset{\frown}{CD} + \overset{\frown}{DB}$.

$\therefore \quad \overset{\frown}{AC} < \overset{\frown}{CB}$.

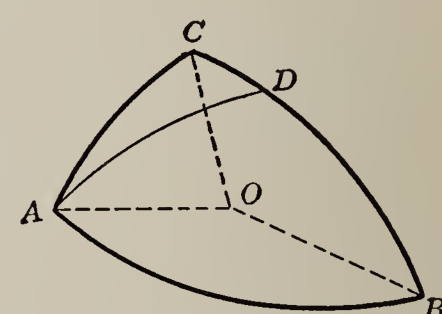


Fig. 229

348. Theorem: *If two sides of a spherical triangle are unequal, the angles opposite are unequal and the greater angle lies opposite the greater side.*

Prove by the indirect method.

349. Theorem: *The diameters of a sphere drawn through the vertices of a given spherical triangle meet the surface of the sphere in points which are the vertices of a triangle symmetrical to the given triangle.*

Prove that ⚠ABC and $A'B'C'$, Fig. 230, are mutually equilateral and therefore mutually equiangular.

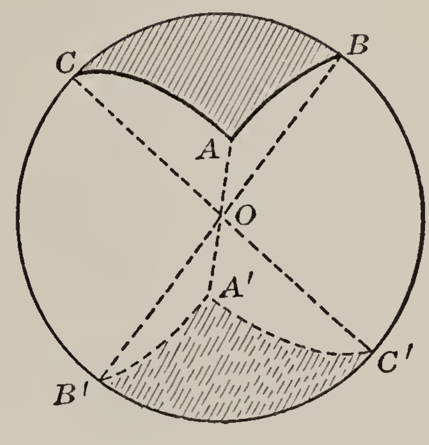


FIG. 230

Area of a Spherical Triangle

350. Theorem: *Two symmetrical spherical triangles are equal.*

Given the symmetrical spherical triangles ABC and $A'B'C'$, Fig. 231.

To prove that

$$\triangle ABC = \triangle A'B'C'.$$

Proof: Let the two triangles be placed so that A and A', B and B', C and C' are opposite endpoints of diameters. Let P be the pole of the small circle determined by points A, B, and C, and let P' be the other endpoint of the diameter through P.

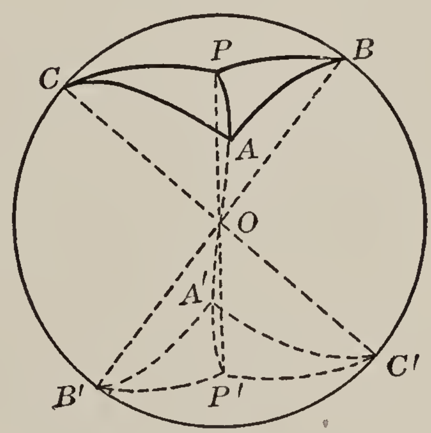


FIG. 231

Show that ⚠APB and $A'P'B'$ are isosceles, symmetrical, and therefore congruent.

Similarly, $\triangle CPA \cong \triangle C'P'A'$

and $\triangle BPC \cong \triangle B'P'C'$.

Adding, $\triangle ABC = \triangle A'B'C'$.

351. Lune. Two great semicircles, ACB and ADB, Fig. 232, form a **lune.** The spherical angle CAD is the **angle of the lune.** The portion of the surface of the sphere included between the semicircles is the **surface of the lune,** and its area is the **area** of the lune.

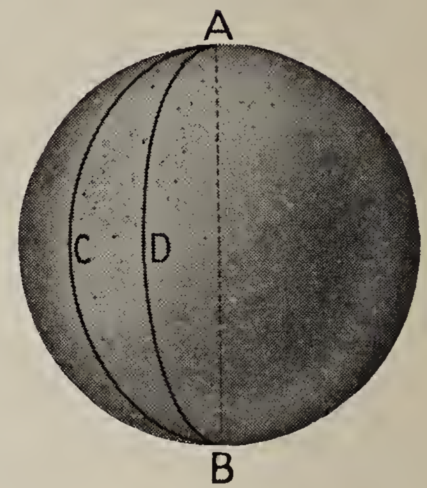


Fig. 232

352. Spherical degree. The area of a spherical triangle having two sides equal to quadrants and the included spherical angle equal to 1° is used as the unit of measure of areas of spherical polygons. It is called a **spherical degree.**

EXERCISES

1. Show that a trirectangular triangle contains 90 spherical degrees.

2. Show that the surface of a sphere contains 720 spherical degrees.

3. *The number of spherical degrees in the surface of a lune is twice the number of degrees in its angle,* i.e.,

$$L = 2(A),$$

where A is the number of degrees in the angle of the lune and L the number of *spherical degrees* in the surface of the lune.

4. Show that the area of a spherical degree, in units of plane surface, is $\frac{\pi R^2}{90}$.

5. Show that the area, S, of a lune, in units of plane surface, is $\frac{\pi R^2 A}{90}$, where A is the number of degrees in the angle of the lune and R the radius of the sphere.

353. Theorem: *The area, in spherical degrees, of a spherical triangle is equal to the spherical excess.*

Given the spherical triangle ABC, Fig. 233.

To prove that the area, S, of $\triangle ABC$ is given by the formula

$$S=(A+B+C-180)$$
spherical degrees,

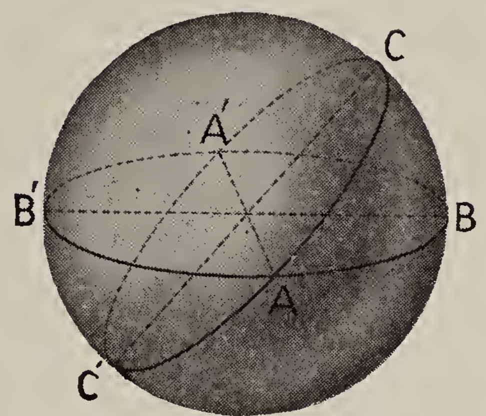


FIG. 233

A, B, and C being the number of degrees in the angles of $\triangle ABC$ and S the number of spherical degrees in $\triangle ABC$.

Proof: Extend the sides of $\triangle ABC$.

The angles of $\triangle ABC$ are the angles of lunes $C'ACBC'$, $B'CBAB'$, and $A'BACA'$ respectively.

Lune $C'ACBC'=\triangle ABC+\triangle ABC'=2(C)$, exercise 3, § 352.

$$\text{Lune } B'CBAB'=\triangle ABC+\triangle CB'A=2(B)$$
$$\text{Lune } A'BACA'=\triangle ABC+\triangle CA'B=2(A)$$

$$3(\triangle ABC)+\triangle ABC'+\triangle CB'A+\triangle CA'B=2(A+B+C)$$
or $$(2\triangle ABC)+\triangle ABC+\triangle ABC'+\triangle CB'A+\triangle CA'B=2(A+B+C)$$

Show that $\triangle CA'B=\triangle C'AB'$

$$\therefore\ 2(\triangle ABC)+\triangle ABC+\triangle ABC'+\triangle CB'A+\triangle C'AB'=2(A+B+C)$$
$$\therefore\ 2(\triangle ABC)+\text{hemisphere}=2(A+B+C)$$
$$\therefore\ 2(\triangle ABC)+\tfrac{1}{2}(720)=2(A+B+C)$$
$$\therefore\ \triangle ABC=A+B+C-\tfrac{1}{4}(720)$$

or $$\boldsymbol{S=A+B+C-180.}$$

354. Let $ABCDE$, Fig. 234, be a spherical polygon of n sides.

Divide $ABCDE$ into $n-2$ spherical triangles by drawing arcs of great circles as AC and AD.

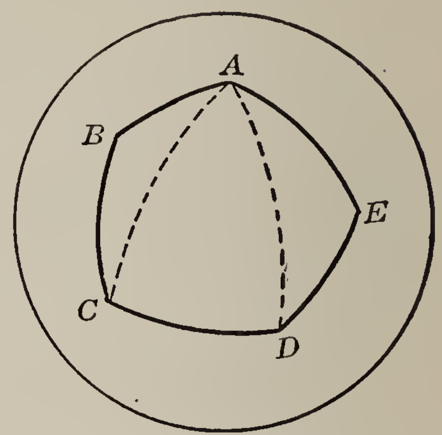


FIG. 234

Denoting by T_1, T_2, T_3, etc., the areas of $\triangle BAC$, CAD, DAE etc., and by s_1, s_2, s_3, etc., the sums of the angles in triangles BAC, CAD, DAE, etc., it follows that

$$T_1 = s_1 - 180,$$
$$T_2 = s_2 - 180,$$
$$T_3 = s_3 - 180, \text{ etc.}$$

Adding, $P = [s_1 + s_2 + s_3 + \text{etc.} - (n-2)180]$,

or $\boldsymbol{P = s - (n-2)180},$

where P denotes the number of spherical degrees in the polygon and s the number of degrees in the sum of the angles.

This may be stated as a theorem as follows:

The area of a spherical polygon is equal to its spherical excess.

EXERCISES

1. The angles of a spherical triangle are 90°, 90°, and 79°. Find the area in spherical degrees.

2. Find the area of a spherical degree on the spheres having radius equal to 3 in.; 14 in.; a inches.

3. Find the area of a spherical polygon whose angles are 70°, 105°, 145°, 125°, 150°.

4. Find the area of a spherical triangle whose angles are 85°, 120°, and 95°, on a sphere whose radius is 6 inches.

5. The area of a spherical triangle is 100 sq. in. and its angles are 100°, 64°, 200°. What is the radius of the sphere on which the triangle lies? (Board.)

6. Prove that the area of a spherical triangle is proportional to its spherical excess.

Three complete great circles drawn on a sphere whose radius is 10 in. divide the surface of the sphere into eight spherical triangles; the angles of one of these triangles are 40°, 80°, and 120°. Find the area in square inches of each of the eight triangles. (Harvard.)

7. On a sphere of radius 2 ft. the area of a certain triangle is 2 square yards. What is the perimeter of the polar triangle? (Harvard.)

Summary

355. The chapter has taught the meaning of the following terms:

polyedral angle, triedral angle
spherical polygon
spherical triangle
similar polyedrons
spherical angle
right, birectangular, and trirectangular spherical triangles
polar spherical triangles
spherical excess
congruent and symmetrical figures
lune
spherical degree

356. The following theorems have been studied:

1. *The sum of two face angles of a triedral angle is greater than the third.*

2. *The sum of two sides of a spherical triangle is greater than the third side.*

3. *The shortest line that can be drawn between two given points on the surface of a sphere is the minor arc of the great circle which passes through the two points.*

4. *The sum of the face angles of a convex polyedral angle is less than four right angles.*

5. *The sum of the sides of a convex spherical polygon is less than 360°.*

6. *There cannot be more than five kinds of regular polyedrons.*

7. *Two tetraedrons having a triedral angle of one equal to a triedral angle of the other are to each other as the products of the edges including the equal triedral angles.*

8. *Two similar polyedrons are to each other as the cubes of two corresponding edges.*

9. *A spherical angle is measured by the arc of a great circle having the vertex as a pole, and included between the sides produced if necessary.*

10. *A spherical angle is equal to the diedral angle formed by the planes of the sides.*

11. *If a spherical triangle is the polar triangle of another, then the second is the polar triangle of the first.*

12. *In two polar spherical triangles each angle of the one is the supplement of that side of the other of which it is the pole.*

13. *The sum of the angles of a spherical triangle is less than six and greater than two right angles.*

14. *If two triedral angles have the corresponding face angles equal, the corresponding diedral angles are equal.*

15. (1) *If the face angles of one triedral angle are equal respectively to the face angles of another;*

(2) *If two face angles and the included diedral angle are equal respectively to the corresponding parts of the other;*

(3) *If two diedral angles and the included face angle are equal respectively to the corresponding parts of the other;*

(4) *If the diedral angles of one are equal respectively to the corresponding parts of the other;*

The triedral angles are congruent if the parts are arranged in the same order, and symmetrical if they are arranged in the reverse order.

16. *Two spherical triangles are congruent, or symmetrical, if they have the following corresponding parts equal:*

(1) *Three sides;* (2) *three angles;* (3) *two sides and the included angle;* (4) *two angles and the included side.*

17. *If two spherical triangles on the same or equal spheres are mutually equiangular they are mutually equilateral, and conversely.*

18. *The base angles of an isosceles spherical triangle are equal, and conversely.*

19. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and conversely.*

20. *The diameters of a sphere drawn through the vertices of a spherical triangle meet the surface of the sphere in points which are the vertices of a triangle symmetrical to the given triangle.*

21. *Two symmetrical spherical triangles are equal.*

357. The following constructions were taught:

1. *To construct a sphere passing through four given points not all in the same plane.*

2. *To inscribe a sphere in a given tetraedron.*

3. *To determine the diameter of a given material sphere.*

358. The following formulas have been proved:

1. The area of a lune

$$L = 2(A) \text{ spherical degrees,}$$

where A is the number of degrees in the angle of the lune.

2. The area of a spherical triangle,

$$T = (A + B + C - 180) \text{ spherical degrees,}$$

where A, B, and C are the number of degrees in the angles of the triangle.

This formula may be stated,

$$T = E, \text{ the spherical excess.}$$

3. The area of a spherical polygon,

$$P = (s - (n-2)180) \text{ spherical degrees,}$$

where s is the number of degrees in the sum of the angles of the polygon.

This formula may be stated,

$$P = E, \text{ the spherical excess.}$$

CHAPTER XV

SUMMARY OF THE ASSUMPTIONS AND THEOREMS OF GEOMETRY GIVEN IN THE COURSES OF THE FIRST AND SECOND YEARS

359. For the convenience of the student a complete list of the assumptions and theorems studied in the first two courses is given below. References in the foregoing chapters are made to this list to save the student the time of looking them up in other textbooks. The numbers in the parentheses ( ) refer to the sections in *First-Year Mathematics*, those in brackets [ ] to the sections in *Second-Year Mathematics* in which these statements were given for the first time.

Preliminary Assumptions

360. Through two points one and only one straight line can be drawn. (20)

361. A straight line two of whose points lie in a plane lies entirely in the plane. (204)

362. The shortest distance between two points is the straight line-segment joining the points. (21)

363. Two straight lines intersect in one and only one point. (25)

364. A line-segment, or an angle, is equal to the sum of all its parts. (33)

365. A segment, or an angle, is greater than any of its parts, if only positive magnitudes are considered. (34)

366. If the same number is added to equal numbers, the sums are equal. (35)

367. If equals are added to equals, the sums are equal. (36)

368. If the same number or equal numbers be subtracted from equal numbers, the differences are equal. (41)

369. The sums obtained by adding unequals to equals are unequal in the same order as are the unequal addends. (42)

370. The sums obtained by adding unequals to unequals in the same order, are unequal in the same order. (43)

371. The differences obtained by subtracting unequals from equals are unequal in the order opposite to that of the subtrahends. (44)

372. If equals be divided by equal numbers (excluding division by 0), the quotients are equal. (78)

373. If equals be multiplied by the same number or equal numbers, the products are equal. (80)

Angles

374. All right angles are equal. (118)

375. Equal central angles in the same or equal circles intercept equal arcs. (124)

376. In the same or equal circles equal arcs are intercepted by equal central angles. (125)

377. A central angle is measured by the intercepted arc. (126)

378. If two angles have their sides parallel respectively they are equal or supplementary. (197)

379. If the sum of two adjacent angles is a straight angle, the exterior sides are in the same straight line. (177)

380. The sum of all the adjacent angles about a point, on one side of a straight line, is a straight angle. (179)

381. The sum of all the angles at a point just covering the angular space about the point is a perigon. (180)

382. If two lines intersect, the opposite angles are equal. (183)

Angles of a Triangle and Polygon

383. The sum of the angles of a triangle is 180°. (112), (198)

384. The sum of the exterior angles of a triangle, taking one at each vertex, is 360°. (115)

385. An exterior angle of a triangle is equal to the sum of the two remote interior angles. (118), (199)

386. If the angles of one triangle are respectively equal to the angles of another, the third angles are equal. (281)

387. The base angles of an isosceles triangle are equal. (280)

388. An equilateral triangle is equiangular. (281)

389. If two angles of a triangle are equal the triangle is isosceles. (281)

390. The acute angles of a right triangle are complementary angles. (184)

391. In a right triangle whose acute angles are 30° and 60°, the side opposite the 90°-angle is twice as long as the side opposite the 30°-angle. (185)

392. The sum of the interior angles of a polygon is $(n-2)$ straight angles. [88]

393. The sum of the exterior angles of a polygon, taking one at each vertex, is 360°. [89]

Perpendicular Lines

394. The shortest distance from a point to a line is the perpendicular from the point to the line. (285)

395. At a given point in a given line one and only one perpendicular can be drawn to the line. (176)

396. From a given point without a straight line one perpendicular can be drawn to the line, and only one. [296]

397. All points on the perpendicular bisector of a line-segment are equidistant from the endpoints of the segment. (281)

398. If a point is equidistant from the endpoints of a line-segment, it is on the perpendicular bisector of the segment. (283)

399. If each of two points on a given line is equally distant from two given points, the given line is the perpendicular bisector of the segment joining the given points. [83]

Parallel Lines

400. Parallel lines are everywhere equally distant. (192)

401. One and only one parallel can be drawn to a line from a point outside the line. (194)

402. If two lines are cut by a transversal making the corresponding angles equal, the lines are parallel. (195)

403. Two lines perpendicular to the same line are parallel. [110]

404. Two lines are parallel if two alternate interior angles formed with a transversal are equal. [112]

405. Two lines are parallel if the interior angles on the same side formed with a transversal are supplementary. [116]

406. Two lines parallel to the same line are parallel to each other. (195)

407. If two parallel lines are cut by a transversal, the corresponding angles are equal; the alternate interior angles are equal; the interior angles on the same side are supplementary. (196)

408. A line perpendicular to one of two parallel lines is perpendicular to the other. [116]

409. A line bisecting a side of a triangle and parallel to a second side bisects the third side. [159]

410. If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal. [160]

411. The line joining the midpoints of two sides of a triangle is parallel to the third side. [168]

Congruent Triangles

412. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other. (*s.a.s.*) (274)

413. Two triangles are congruent if two angles and the side included between their vertices in one are equal respectively to the corresponding parts in the other. (*a.s.a.*) (275)

414. If three sides of one triangle are equal, respectively, to the three sides of another triangle, the triangles are congruent. (*s.s.s.*) (283)

415. Two *right* triangles are congruent if the hypotenuse and one side of one are equal respectively to the hypotenuse and a side of the other. (285)

Quadrilaterals

416. If a quadrilateral is a parallelogram—

1. A diagonal divides it into congruent triangles;
2. The opposite sides are equal;
3. The opposite angles are equal;
4. The consecutive angles are supplementary;
5. The diagonals bisect each other. [118–122]

417. A quadrilateral is a parallelogram if—

1. The opposite sides are parallel;
2. The opposite sides are equal;
3. One pair of opposite sides are equal and parallel;
4. The opposite angles are equal;
5. The diagonals bisect each other. [123–127]

Similar Figures

418. A line parallel to one side of a triangle forms with the other two sides a triangle similar to the given triangle. [214]

419. Two triangles are similar if two angles of one are respectively equal to two angles of the other. [217]

420. Two triangles are similar if the ratio of two sides of one equals the ratio of two sides of the other and the angles included between these sides are equal. [218]

421. Two triangles are similar if the corresponding sides are in proportion. [219]

422. The perimeters of similar polygons are to each other as any two homologous sides. [219]

423. Similar polygons may be divided by homologous diagonals into triangles similar to each other and similarly placed. [220]

424. The perpendicular to the hypotenuse from the vertex of the right angle divides a right triangle into parts similar to each other and to the given triangle. [224]

Relations between the Sides of a Triangle

425. In a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides of the right angle. [233], algebraic proof; [462], geometric proof.

426. In a triangle the square on the side opposite an *acute* angle is equal to the sum of the squares of the other two sides *diminished* by two times the product of one of these two sides and the projection of the other upon it. [240]

427. In a triangle the square on the side opposite the *obtuse* angle is equal to the sum of the squares on the other two sides, *increased* by two times the product of one of them and the projection of the other upon it. [241]

428. In a triangle the sum of the squares of two sides is equal to twice the square of one-half of the third side increased by twice the square of the median to the third side.

Proportional Line-Segments

429. The perimeters of similar polygons are in the same ratio as any two corresponding sides. [219]

430. If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other. [314]

431. If from a point without a circle a tangent and secant be drawn, the tangent is a mean proportional between the entire secant to the concave arc and the external segment. [315]

432. If from a point without a circle two secants be drawn to the concave arc, the product of one secant and its external segment is equal to the product of the other secant and its external segment. [317]

433. In a right triangle the perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse. [230]

434. In a right triangle either side of the right angle is the mean proportional between its projection upon the hypotenuse and the entire hypotenuse. [232]

435. A perpendicular to a diameter of a circle at any point, extended to the circle, is the mean proportional between the segments of the diameter. [231]

436. If two parallels cut two intersecting transversals, the segments intercepted on one transversal are proportional to the corresponding segments on the other. [163]

437. If a number of parallels cut two transversals, the segments intercepted on one transversal are proportional to the corresponding segments on the other. [167]

438. Two lines that cut two given intersecting lines and make the corresponding segments of the given lines proportional, are parallel. [167]

439. The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides. [170]

Circles

440. A point is within, upon, or without, a circle according as its distance from the center is less than, equal to, or greater than, the radius. [274]

441. Circles having equal radii are equal, and equal circles have equal radii. [275]

442. A diameter divides a circle into equal parts. [276]

443. The radius drawn to the point of contact of a tangent is perpendicular to the tangent. (308)

444. A line perpendicular to a radius at the outer endpoint is tangent to the circle. (309)

445. A circle can be drawn through three points not in a straight line. (312)

446. In the same or equal circles equal central angles intercept equal arcs, and equal arcs are intercepted by equal central angles. [281]

447. In the same or equal circles equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs. [283]

448. If two tangents to the same circle intersect, the distances from the point of intersection to the points of contact are equal. [315]

449. If any two of the following five conditions are taken as hypothesis, the remaining three are true:

1. A line passes through the center.
2. A line is perpendicular to a chord.
3. A chord is bisected by a line.
4. A minor arc is bisected.
5. A major arc is bisected. [284]

450. In the same or equal circles equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal. [285]

451. The arcs included between two parallel secants are equal; and, conversely, if two secants include equal arcs and do not intersect within the circle, they are parallel. [286]

452. The line joining the centers of two intersecting circles bisects the common chord perpendicularly. [287]

453. If two circles are tangent to each other, the centers and the point of tangency lie in a straight line. [289]

Measurement of Angles by Arcs

454. A central angle is measured by the intercepted arc. [297]

455. In the same or equal circles two central angles have the same ratio as the intercepted arcs. [297]

456. An inscribed angle is measured by one-half the arc intercepted by the sides. [298]

457. An angle formed by a tangent and a chord passing through the point of contact is measured by one-half of the intercepted arc. [300]

458. If two chords intersect within a circle, either angle formed is measured by one-half the sum of the intercepted arcs. [304]

459. If two secants meet outside of a circle, the angle formed is measured by one-half the difference of the intercepted arcs. [305]

460. The angle formed by a tangent and a secant meeting outside of a circle is measured by one-half the difference of the intercepted arcs. [306]

461. The angle formed by two tangents to a circle is equal to one-half the difference of the intercepted arcs. [307]

Regular Polygons and the Circle

462. If a circle is divided into equal arcs, the chords subtending these arcs form a regular inscribed polygon. [435]

463. If the midpoints of the arcs subtended by the sides of a regular inscribed polygon of n sides are joined to the adjacent vertices of the polygon, a regular inscribed polygon of $2n$ sides is formed. [436]

464. If a circle is divided into equal arcs, the tangents drawn at the points of division form a regular circumscribed polygon. [437]

465. If tangents are drawn to a circle at the midpoints of the arcs terminated by consecutive points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon is formed having double the number of sides. [438]

466. If at the midpoints of the arcs subtended by the sides of a given regular inscribed polygon, tangents are drawn to the circle, they are parallel to the sides of the given polygon and form a regular circumscribed polygon. [446]

467. A circle may be circumscribed about any given regular polygon. [445]

468. A circle may be inscribed in any given regular polygon. [446]

469. The perimeter of a regular inscribed $2n$-side is greater than the perimeter of the regular n-side inscribed in the same circle. [447]

470. The perimeter of a regular circumscribed $2n$-side is less than the perimeter of the regular n-side circumscribed about the same circle. [448]

471. The length of a circle is expressed by the formula

$$C = \pi d, \text{ or } C = 2\pi r. \quad [450]$$

Inequalities

472. The diameter of a circle is larger than any other chord of the circle. [339]

473. An exterior angle of a triangle is greater than either of the remote interior angles. [339]

474. If two sides of a triangle are unequal, the angles opposite to them are unequal, the greater angle lying opposite the greater side. (281)

475. If two angles of a triangle are unequal the sides opposite to them are unequal, the greater side lying opposite the greater angle. (281)

476. If two oblique line-segments drawn to a line from a point on a perpendicular to the line have unequal projections, the oblique line-segments are unequal. [340]

477. Two unequal oblique line-segments drawn to a line from a point on a perpendicular to the line have unequal projections. [341]

478. If from a point inside a triangle line-segments are drawn to the endpoints of one side, the sum of these line-segments is less than the sum of the other two sides. [348]

479. In the same, or in equal, circles unequal chords are unequally distant from the center, the shorter chord lying at the greater distance; and the converse of this theorem. [349]

480. If two sides of one triangle are equal to two sides of another triangle, but the angle included between the two sides of the first is greater than the angle included between the corresponding sides in the second, then the third side in the first is greater than the third side in the second; and the converse of this theorem. [350]

481. In the same, or in equal, circles the arcs subtended by unequal chords are unequal in the same order as the chords; and the converse of this theorem. [351]

Loci. Concurrent Lines

482. The locus of points in a plane equidistant from two given points is the perpendicular bisector of the segment joining these points. [407]

483. The locus of points in a plane which are within an angle and equidistant from its sides is the bisector of the angle. [408].

484. The locus of points in a plane at a given distance from a given point is the circle whose center is the given point and whose radius is equal to the given distance. [409]

485. The locus of points in a plane at a given distance from a given line consists of a pair of lines parallel to the given line and the given distance from it. [410]

486. The locus of points in space equidistant from all points on a circle is the line perpendicular to the plane of the circle at the center. [411]

487. The locus of points in space equidistant from two given points is the plane bisecting the segment joining these points and perpendicular to it. [412]

488. The locus of points within a diedral angle equidistant from the faces is the plane bisecting the angle. [413]

Concurrent Lines

489. The medians of a triangle are concurrent. [417]

490. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle. [419]

491. The bisectors of the angles of a triangle are concurrent in a point which is equidistant from the sides of the triangle. [421]

492. The three altitudes of a triangle are concurrent. [422]

Areas

493. Parallelograms having equal bases and equal altitudes are equal. [459]

494. The area of a rectangle is equal to the product of the base and the altitude,

$$R = b \cdot h. \quad [481]$$

495. The area of a parallelogram is equal to the product of the base and the altitude,

$$P = b \cdot h. \quad [482]$$

496. The area of a triangle is equal to one-half the product of the base and altitude,

$$A = \tfrac{1}{2} b \cdot h. \quad [465]$$

497. The area of a triangle is equal to one-half the product of two sides by the sine of the included angle,

$$A = \tfrac{1}{2} ab \sin C. \quad [466]$$

498. The area of a triangle is equal to one-half the perimeter times the radius of the inscribed circle,

$$A = \tfrac{1}{2} p \cdot r = sr. \quad [467]$$

499. The area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle,

$$A = \frac{abc}{4r}. \quad [468]$$

500. The area of a triangle is equal to

$$A = \sqrt{s(s-a)(s-b)(s-c)}. \quad [469]$$

501. The area of an equilateral triangle is one-fourth the square of a side times the square root of 3,

$$A = \frac{a^2}{4}\sqrt{3}. \quad [471]$$

502. The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases,

$$T = \tfrac{1}{2} h(b_1 + b_2). \quad [483]$$

503. The area of a regular inscribed polygon is equal to the product of one-half the perimeter and the apothem. [484]

504. The area of a regular circumscribed polygon is equal to the product of one-half the perimeter and the radius. [485]

505. The area of a circle is one-half the product of the length of the circle and the radius, i.e.,

$$A = \tfrac{1}{2}cr. \quad [489]$$

506. The area of a circle is given by the formula

$$A = \pi r^2. \quad [489]$$

507. The area of a sector is given by the formula

$$A = \tfrac{1}{2}a'r. \quad [490]$$

508. The area of a segment of a circle is given by the formulas

$$A = \frac{1}{2}a'r - \frac{1}{2}a\sqrt{r^2 - \frac{a^2}{4}},$$

or

$$A = \frac{1}{2}a'r - \frac{1}{2}r^2 \sin x. \quad [491]$$

Proportionality of Areas

509. In a proportion the product of the means is equal to the product of the extremes. (259)

510. The areas of two rectangles are in the same ratio as the products of their dimensions. (260)

511. Two rectangles having equal bases are in the same ratio as the altitudes. (261)

512. Two rectangles having equal altitudes are in the same ratio as the bases. (262)

513. The areas of parallelograms are in the same ratio as the products of the bases and altitudes. (263)

514. The areas of triangles are in the same ratio as the products of the bases and altitudes. (264)

515. The areas of parallelograms having equal bases are in the same ratio as the altitudes. (265)

516. The areas of triangles having equal bases are in the same ratio as the altitudes. (266)

517. The areas of two similar triangles are to each other as the squares of any two corresponding sides. [497]

518. The areas of two similar polygons are to each other as the squares of two corresponding sides. [498]

Lines and Planes in Space

519. The following conditions determine the position of a plane in space:

1. A straight line and a point not in that line.
2. Three points not in the same straight line.
3. Two intersecting straight lines.
4. Two parallel straight lines. [139]

520. If two planes intersect, the intersection is a straight line. [143]

521. Two planes perpendicular to the same line are parallel. [178]

522. If two parallel planes are cut by a third plane, the intersections are parallel. [179]

523. Parallel line-segments intercepted by parallel planes are equal. [180]

524. If three or more parallel planes are cut by two transversals, the corresponding segments of the transversals are in proportion. [181]

525. The projection upon a plane, of a straight line not perpendicular to the plane, is a straight line. [355]

526. The projection upon a plane, of a straight line perpendicular to the plane, is a point. [356]

527. The acute angle formed by a given line and its projection upon a plane is smaller than the angle which it makes with any other line in the plane passing through the point of intersection of the given line and the plane. [357]

528. The perpendicular is the shortest distance from a point to a plane. [342]

529. Oblique lines drawn from a point to a plane, meeting the plane at points equidistant from the foot of the perpendicular, are equal. [344]

530. Oblique lines drawn from a point to a plane, meeting the plane at points unequally distant from the foot of the perpendicular, are unequal, the more remote being the greater. [345]

531. Equal oblique lines drawn from a point to a plane meet the plane at points equidistant from the foot of the perpendicular. [346]

532. Of two unequal oblique lines drawn from a point to a plane the greater meets the plane at the greater distance from the foot of the perpendicular. [347]

533. If a line is perpendicular to each of two intersecting lines it is perpendicular to the plane determined by these lines. [364]

534. All the perpendiculars to a given line at a given point lie in a plane perpendicular to the given line at the point. [366]

535. Only one plane can be constructed perpendicular to a given line at a given point. [367]

536. Only one plane can be constructed perpendicular to a given line from a point outside of the line. [368]

537. Only one line can be constructed perpendicular to a given plane at a given point. [370]

538. From a point outside of a given plane only one line can be constructed perpendicular to the plane. [372]

539. Lines perpendicular to a plane are parallel. [373]

540. If one of two parallel lines is perpendicular to a plane, the other is perpendicular to the same plane. [374]

541. Two lines parallel to the same line are parallel to each other. [375]

542. If two lines are parallel, a plane containing one of them and not the other, is parallel to the other. [376]

543. If one of two parallel planes is perpendicular to a line the other is also. [377]

544. If two intersecting lines are parallel to a given plane, their plane is parallel to the given plane. [378]

545. If two angles not in the same plane have their sides parallel and running in the same direction, the angles are equal and their planes are parallel. [379]

546. All plane angles of a diedral angle are equal. [380]

547. If two diedral angles are equal their plane angles are equal. [381]

548. Two diedral angles are equal if the plane angles are equal. [381]

549. If a line is perpendicular to a plane every plane through this line is perpendicular to the plane. [382]

550. If two planes are perpendicular to each other a line drawn in one of them perpendicular to the intersection is perpendicular to the other. [383]

551. If two planes are perpendicular to each other a line perpendicular to one of them at a point of the intersection must lie in the other. [383]

552. If from a point in one of two perpendicular planes a line is drawn perpendicular to the other, it must lie in the first plane. [383]

553. If a plane is perpendicular to each of two planes it is perpendicular to their intersection. [384]

554. Through a line not perpendicular to a given plane, one plane and only one may be passed perpendicular to the given plane. [385]

555. The section of a sphere made by a plane is a circle. [389]

556. The axis of a circle passes through the center. [390]

557. The diameter of a sphere passing through the center of a circle is perpendicular to the plane of the circle. [390]

558. All great circles of a sphere are equal. [390]

559. Every great circle bisects the surface of the sphere. [390]

560. Through two points on the surface of a sphere, not the endpoints of a diameter, only one great circle can be drawn. [390]

561. All points on a circle of a sphere are equidistant from its poles. [392]

562. The polar distance of a great circle is a quadrant. [395]

563. If a point on the surface of a sphere is at the distance of a quadrant from each of two given points on the surface, it is a pole of the great circle passing through the given points. [396]

564. The intersection of the surfaces of two spheres is a circle whose plane is perpendicular to the line of centers of the spheres, and whose center is in that line. [397]

565. A plane tangent to a sphere is perpendicular to the radius at the point of contact. [399]

566. A plane perpendicular to a radius of a sphere at the outer extremity is tangent to the sphere. [400]

Constructions

567. Through a given point in a given line pass a plane perpendicular to a given line. [365]

568. From a given point outside of a given line construct a plane perpendicular to the given line. [368]

569. At a given point in a given plane construct a perpendicular to the plane. [369]

570. From a point outside of a plane construct a line perpendicular to the plane. [371]

571. To pass a plane perpendicular to a given plane, that shall contain a line not perpendicular to the given plane. [385]

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474

LOGARITHMS OF NUMBERS—*Continued*

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	9525	8531	8537	8543	8549	8555	9561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	9698	8704	9710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0008	0013	0017	0021	0026	0030	0035	0039

TABLE OF POWERS AND ROOTS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7.056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	**90**	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.641

TABLE OF SINES, COSINES, AND TANGENTS OF ANGLES FROM 1°–90°

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	.0872	.9962	.0875	**50**	.7660	.6428	1.1918
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	.1736	.9848	.1763	**55**	.8192	.5736	1.4281
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	.2588	.9659	.2679	**60**	.8660	.5000	1.7321
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.9888	.4384	2.0503
20	.3420	.9397	.3640	**65**	.9063	.4226	2.1445
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	.4226	.9063	.4663	**70**	.9397	.3420	2.7475
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	.5000	.8660	.5774	**75**	.9659	.2588	3.7321
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	.5736	.8192	.7002	**80**	.9848	.1736	5.6713
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	.6428	.7660	.8391	**85**	.9962	.0872	11.4301
41	.6561	.7547	.8693	86	.9976	.0698	14.3006
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	.7071	.7071	1.0000	**90**	1.0000	.0000	∞

Formulas

PLANE GEOMETRY

1. Length of circle $= 2\pi r = 3.14159d$
2. Area of circle $= \pi r^2$
3. Area of triangle $= \frac{1}{2}bh = \frac{1}{2}ab \sin C = \frac{1}{2}r(a+b+c)$
   $= \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$
4. Area of parallelogram $= bh$
5. Area of square $= a^2$
6. Area of equilateral triangle $= \frac{a^2}{4}\sqrt{3}$
7. Area of trapezoid $= \frac{1}{2}h(b_1+b_2) = hm$

SOLID GEOMETRY

1. Volume of prism $= ba$
2. Volume of pyramid $= \frac{1}{3}ba$
3. Volume of right circular cylinder $= \pi r^2 a$
4. Total surface of right circular cylinder $= 2\pi r(r+a)$
5. Lateral surface of right circular cylinder $= 2\pi ra$
6. Volume of right circular cone $= \frac{1}{3}\pi r^2 a$
7. Lateral surface of right circular cone $= \pi rs$
8. Total surface of right circular cone $= \pi r(r+s)$
9. Surface of sphere $= 4\pi r^2$
10. Volume of sphere $= \frac{4}{3}\pi r^3$

SERIES

1. Arithmetical progression:

$$l = a + (n-1)d; \quad s = \frac{n}{2}(a+l)$$

2. Geometrical progression:

$$l = ar^{n-1};\quad s = \frac{a - ar^n}{1-r};\quad \text{if } r < 1 \text{ and } n \to \infty,\ s = \frac{a}{1-r}$$

3. Binomial theorem:

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \text{etc.}$$

The kth term

$$= \frac{n(n-1)(n-2)\ \ldots\ (n-k+2)}{1 \cdot 2 \cdot 3\ \ldots\ k-1}a^{n-k+1}b^{k-1}$$

LOGARITHMS

1. $\log ab = \log a + \log b$

2. $\log \frac{a}{b} = \log a - \log b$

3. $\log a^n = n \log a$

4. $\log \sqrt[n]{a} = \frac{\log a}{n}$

5. $\log 1 = 0$

6. $\log_a N = \frac{\log_b N}{\log_b a}$

7. $\text{colog } N = \log \frac{1}{N} = (10 - \log N) - 10$

QUADRATIC EQUATION

If $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac = 0$, the roots are real and equal.

If $b^2 - 4ac > 0$, the roots are real and unequal.

If $b^2 - 4ac < 0$, the roots are complex.

TRIGONOMETRIC FORMULAS

1. $\sin \alpha = \frac{a}{c}$, $\cos \alpha = \frac{b}{c}$, $\tan \alpha = \frac{a}{b}$,

   $\csc \alpha = \frac{c}{a}$, $\sec \alpha = \frac{c}{b}$, $\cot \alpha = \frac{b}{a}$

2. $\sin^2 \alpha + \cos^2 \alpha = 1$
3. $\sec^2 \alpha = 1 + \tan^2 \alpha$
4. $\csc^2 \alpha = 1 + \cot^2 \alpha$
5. $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$
6. $\cot \alpha = \frac{\cos \alpha}{\sin \alpha}$
7. $\sec \alpha = \frac{1}{\cos \alpha}$
8. $\csc \alpha = \frac{1}{\sin \alpha}$

9. $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$
10. $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
11. $\tan(\alpha \pm \beta) = \dfrac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$

12. $\sin \alpha + \sin \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$
13. $\sin \alpha - \sin \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)$
14. $\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$
15. $\cos \alpha - \cos \beta = -2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)$

16. $\sin \alpha \sin \beta = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta)$
17. $\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta)$
18. $\sin \alpha \cos \beta = \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta)$

19. $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
20. $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

    $= 2\cos^2 \alpha - 1 = 1 - 2\sin^2 \alpha$

21. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

22. $\sin\frac{1}{2}\alpha = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

23. $\cos\frac{1}{2}\alpha = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

24. $\tan\frac{1}{2}\alpha = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

25. $\sin\left(\dfrac{\pi}{2}\pm\theta\right) = \cos\theta$

26. $\cos\left(\dfrac{\pi}{2}\pm\theta\right) = \mp\sin\theta$

27. $\tan\left(\dfrac{\pi}{2}\pm\theta\right) = \mp\cot\theta$

28. $\sin(\pi\pm\theta) = \mp\sin\theta$

29. $\cos(\pi\pm\theta) = -\cos\theta$

30. $\tan(\pi\pm\theta) = \pm\tan\theta$

31. $\sin(-x) = -\sin x$

32. $\cos(-x) = \cos x$

33. $\tan(-x) = -\tan x$

34. $\csc(-x) = -\csc x$

35. $\sec(-x) = \sec x$

36. $\cot(-x) = -\cot x$

37. $\sin 30° = \frac{1}{2}$

38. $\sin 45° = \frac{1}{2}\sqrt{2}$

39. $\sin 60° = \frac{1}{2}\sqrt{3}$

40. $\cos 30° = \frac{1}{2}\sqrt{3}$

41. $\cos 45° = \frac{1}{2}\sqrt{2}$

42. $\cos 60° = \frac{1}{2}$

TRIANGLES

43. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

44. $a^2 = b^2 + c^2 - 2bc\cos A$

45. $\dfrac{a+b}{a-b} = \dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$

46. $\dfrac{a-b}{c} = \dfrac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$

47. $\dfrac{a+b}{c} = \dfrac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}$

If $s = \frac{1}{2}(a+b+c)$:

48. $\sin\frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$.

49. $\cos\frac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}}$.

50. $\tan\frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

If r = radius of inscribed circle:

51. $r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

52. $\tan\frac{1}{2}A=\frac{r}{s-a}$.

53. $\tan\frac{1}{2}B=\frac{r}{s-b}$.

54. $\tan\frac{1}{2}C=\frac{r}{s-c}$.

55. Area $=\frac{1}{2}ab\sin C=\frac{c^2}{2}\cdot\frac{\sin A\sin B}{\sin C}$

$$=\sqrt{s(s-a)(s-b)(s-c)}$$

56. Diameter of circumscribed circle

$$=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Reduction of Degrees, Minutes, and Seconds to Radians

°	Radians	′	Radians	″	Radians
1	0.01 745 33	1	0.00 029 09	1	0.00 000 48
2	0.03 490 66	2	0.00 058 18	2	0.00 000 97
3	0.05 235 99	3	0.00 087 27	3	0.00 001 45
4	0.06 981 32	4	0.00 116 36	4	0.00 001 94
5	0.08 726 65	5	0.00 145 44	5	0.00 002 42
6	0.10 471 98	6	0.00 174 53	6	0.00 002 91
7	0.12 217 30	7	0.00 203 62	7	0.00 003 39
8	0.13 962 63	8	0.00 232 71	8	0.00 003 88
9	0.15 707 96	9	0.00 261 80	9	0.00 004 36
10	0.17 453 29	10	0.00 290 89	10	0.00 004 85
20	0.34 906 59	15	0.00 436 33	15	0.00 007 27
30	0.52 359 88	20	0.00 581 78	20	0.00 009 70
40	0.69 813 17	25	0.00 727 22	25	0.00 012 12
50	0.87 266 46	30	0.00 872 66	30	0.00 014 54
60	1.04 719 76	35	0.01 018 11	35	0.00 016 97
70	1.22 173 05	40	0.01 163 55	40	0.00 019 39
80	1.39 626 34	50	0.01 454 44	50	0.00 024 24
90	1.57 079 63	60	0.01 745 33	60	0.00 029 09

Reduction of Minutes to Degrees

0′=0°.000	**10′**=0°.166	**20′**=0°.333	**30′**=0°.500	**40′**=0°.666	**50′**=0°.833
1′ .016	11′ .183	21′ .350	31′ .516	41′ .683	51′ .850
2′ .033	12′ .200	22′ .366	32′ .533	42′ .700	52′ .866
3′ .050	13′ .216	23′ .383	33′ .550	43′ .716	53′ .883
4′ .066	14′ .233	24′ .400	34′ .566	44′ .733	54′ .900
5′ .083	**15′** .250	**25′** .416	**35′** .583	**45′** .750	**55′** .916
6′ .100	16′ .266	26′ .433	36′ .600	46′ .766	56′ .933
7′ .116	17′ .283	27′ .450	37′ .616	47′ .783	57′ .950
8′ .133	18′ .300	28′ .466	38′ .633	48′ .800	58′ .966
9′ .150	19′ .316	29′ .483	39′ .650	49′ .816	59′ .983
		30′=0°.500			**60′**=1°.000

Reduction of Seconds to Degrees

6″ =0°.00166	10″ =0°.00277	35″ =0°.00970
7″ =0.00194	15″ =0.00416	40″ =0.01111
8″ =0.00222	20″ =0.00555	45″ =0.01250
9″ =0°.0025C	30″ =0°.00833	50″ =0°.01388

Reduction of Degrees to Minutes and Seconds

0°.00 =	0′	**0°.30** =	18′	**0°.60** =	36′	**0°.90** =	54′
.01	0′36″	.31	18′36″	.61	36′36″	.91	54′36″
.02	1′12″	.32	19′12″	.62	37′12″	.92	55′12″
.03	1′48″	.33	19′48″	.63	37′48″	.93	55′48″
.04	2′24″	.34	20′24″	.64	38′24″	.94	56′24″
0°.05	3′	**0°.35** =	21′	**0°.65** =	39′	**0°.95** =	57′
.06	3′36″	.36	21′36″	.66	39′36″	.96	57′36″
.07	4′12″	.37	22′12″	.67	40′12″	.97	58′12″
.08	4′48″	.38	22′48″	.68	40′48″	.98	58′48″
.09	5′24″	.39	23′24″	.69	41′24″	.99	59′24″
0°.10 =	6′	**0°.40** =	24′	**0°.70** =	42′	**1°.00** =	60′
.11	6′36″	.41	24′36″	.71	42′36″		
.12	7′12″	.42	25′12″	.72	43′12″		
.13	7′48″	.43	25′48″	.73	43′48″		
.14	8′24″	.44	26′24″	.74	44′24″		
0°.15 =	9′	**0°.45** =	27′	**0°.75** =	45′		
.16	9′36″	.46	27′36″	.76	45′36″	**0°.000** =	0″
.17	10′12″	.47	28′12″	.77	46′12″	.001	3″.6
.18	10′48″	.48	28′48″	.78	46′48″	.002	7″.2
.19	11′24″	.49	29′24″	.79	47′24″	.003	10″.8
						.004	14″.4
0°.20 =	12′	**0°.50** =	30′	**0°.80** =	48′		
.21	12′36″	.51	30′36″	.81	48′36″	**0°.005** =	18″
.22	13′12″	.52	31′12″	.82	49′12″	.006	21″.6
.23	13′48″	.53	31′48″	.83	49′48″	.007	25″.2
.24	14′24″	.54	32′24″	.84	50′24″	.008	28″.8
						.009	32″.4
0°.25 =	15′	**0°.55** =	33′	**0°.85** =	51′		
.26	15′36″	.56	33′36″	.86	51′36″	**0°.01** =	36″
.27	16′12″	.57	34′12″	.87	52′12″		
.28	16′48″	.58	34′48″	.88	52′48″		
.29	17′24″	.59	35′24″	.89	53′24″		
0°.30 =	18′	**0°.60** =	36′	**0°.90** =	54′		

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